The Astrophysical Journal, 648:L71–L74, 2006 September 1

MAGNETIC HELICITY DENSITY AND ITS FLUX IN WEAKLY INHOMOGENEOUS TURBULENCE

Kandaswamy Subramanian[1] and Axel Brandenburg[2]
Received 2006 June 19; accepted 2006 July 20; published 2006 August 18

ABSTRACT

A gauge-invariant and hence physically meaningful definition of magnetic helicity density for random fields is proposed, using the Gauss linking formula, as the density of correlated field line linkages. This definition is applied to the random small-scale field in weakly inhomogeneous turbulence, whose correlation length is small compared with the scale on which the turbulence varies. For inhomogeneous systems, with or without boundaries, our technique then allows one to study the local magnetic helicity density evolution in a gauge-independent fashion, which was not possible earlier. This evolution equation is governed by local sources (owing to the mean field) and by the divergence of a magnetic helicity flux density. The role of magnetic helicity fluxes in alleviating catastrophic quenching of mean field dynamos is discussed.

Subject headings: MHD — Sun: magnetic fields — turbulence

1. INTRODUCTION

Large-scale magnetic fields produced by dynamo action tend to have some degree of magnetic helicity. A simple example is the interlocking of poloidal and toroidal fields in one hemisphere of a star, seen from stellar dynamo models. However, in the case of the Sun, there is explicit evidence of magnetic helicity being present in or coming from active regions (Seehafer 1990; Pevtsov et al. 1995; Bao et al. 1999; Berger & Ruzmaikin 2000), coronal mass ejections (Démoulin et al. 2002), and the solar wind (Matthaeus et al. 1982; Lynch et al. 2005). While the investigation of magnetic helicity in the Sun and in the solar wind is interesting in its own right, there is now also a direct connection with dynamo theory with the realization that large-scale dynamos must transport and shed small-scale magnetic helicity in order to operate on a dynamical timescale rather than the much longer resistive time-scale (see the review of Brandenburg & Subramanian 2005a for references). A major difficulty with this picture is the absence of a meaningful definition for magnetic helicity *density*, even for small-scale fields. Magnetic helicity is a volume integral, usually defined as $H_M = \int \boldsymbol{A} \cdot \boldsymbol{B}\, dV$, where $\boldsymbol{A}$ is the vector potential and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ is the magnetic field. However, under a gauge transformation $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$, which leaves $\boldsymbol{B}$ invariant, one has $H' = H + \int \Lambda \boldsymbol{B} \cdot d\boldsymbol{S}$. So H is only gauge-invariant if the $\boldsymbol{B}$-field has no component normal to the boundary or if it vanishes sufficiently rapidly at the boundary of the integration volume. In most practical contexts, like the Sun or galaxies, however, the field does not vanish on the boundaries. A possible remedy might be to consider instead the gauge-invariant *relative* magnetic helicity, defined by subtracting the helicity of a reference vacuum field in the same gauge (Berger & Field 1984; Finn & Antonsen 1985). But the flux of relative helicity is cumbersome to work with for arbitrarily shaped boundaries. Also, the concept of a density of relative helicity is not meaningful, since it is defined only as a volume integral. Indeed, there is simply no way that the quantity $\boldsymbol{A} \cdot \boldsymbol{B}$ itself can be gauge-invariant.

On an earlier occasion, Subramanian & Brandenburg (2004, hereafter SB04) considered the evolution of the current helicity density, $H_C = \boldsymbol{J} \cdot \boldsymbol{B}$, where $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}/\mu_0$ is the current density and μ_0 is the vacuum permeability (we set $\mu_0 = 1$ in what follows). Note that H_C as well as its flux are locally well defined, explicitly gauge-invariant, and observationally measurable. Furthermore, from a closure model, Pouquet et al. (1976) show that the α-effect needed for large-scale dynamos has a nonlinear addition due to the small-scale contribution to H_C. The buildup of this small-scale current helicity then goes to cancel the kinetic part of the α-effect and causes catastrophic quenching of the dynamo, unless one can have a helicity flux out of the system. This formed the motivation for the work of SB04. The major disadvantage in working with H_C, however, is that one loses the conceptually simple form of the magnetic helicity conservation law. We propose here instead an alternative means to define magnetic helicity density for the random small-scale field, using the more basic Gauss linking formula for helicity, which can be directly applied to discuss magnetic helicity density and its flux even in inhomogeneous systems with boundaries. The technique applied in calculating the magnetic helicity evolution, however, is very similar to that employed in SB04.

In the following, we define random small-scale quantities as departures from the corresponding mean field quantity, e.g., $\boldsymbol{b} = \boldsymbol{B} - \overline{\boldsymbol{B}}$ for the magnetic field, $\boldsymbol{j} = \boldsymbol{J} - \overline{\boldsymbol{J}}$ for the current density, and $\boldsymbol{u} = \boldsymbol{U} - \overline{\boldsymbol{U}}$ for the velocity. Throughout this Letter, we adopt ensemble averages that, in practice, are commonly approximated as spatial averages over one or two coordinate directions (see, e.g., Brandenburg & Subramanian 2005a). However, the approach developed below applies also to the case without a mean magnetic field. Therefore, specific applications to the mean field dynamo (MFD) will be postponed until the end of this Letter.

2. MAGNETIC HELICITY DENSITY

Given the random small-scale magnetic field $\boldsymbol{b}(\boldsymbol{x}, t)$, one can also define the magnetic helicity directly in terms of the field, as the linkage of its flux, using Gauss's linking formula (Berger & Field 1984; Moffatt 1969):

$$h_{\rm G} = \frac{1}{4\pi} \int \int \boldsymbol{b}(\boldsymbol{x}) \cdot \left[\boldsymbol{b}(\boldsymbol{y}) \times \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^3} \right] d^3x\, d^3y, \qquad (1)$$

[1] Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India; kandu@iucaa.ernet.in.

[2] Nordic Institute for Theoretical Physics (NORDITA), Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark; brandenb@nordita.dk.

where both integrations extend over the full volume. Suppose we define an auxiliary field

$$\boldsymbol{a}(\boldsymbol{x}) = \frac{1}{4\pi}\int \boldsymbol{b}(\boldsymbol{y}) \times \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\, d^3y, \tag{2}$$

then this field satisfies $\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{b}$, and $\boldsymbol{\nabla}\cdot\boldsymbol{a} = 0$, and one can write $h_{\rm G} = \int \boldsymbol{a}\cdot\boldsymbol{b}\, d^3x$. This is the origin of the textbook definition of magnetic helicity in what is known as the Coulomb gauge for the vector potential. Provided the field is closed over the integration volume, this definition can be applied in any other gauge. Note, however, that for an open system with boundaries, it is *not* useful to go to the definition involving the vector potential, which is now of course gauge-dependent. We therefore take the point of view here that the magnetic helicity $h_{\rm G}$ defined by equation (1) is the more basic definition of the topological property determining the links associated with magnetic fields, and not the definition in terms of the vector potential. We will see below that this then also allows us to define naturally a gauge-invariant magnetic helicity density for random fields, as long as the correlation scale of the field is much smaller than the size of the system, as the density of correlated links of the field.

For this, consider $\boldsymbol{b}$ to be a random field with a correlation function $\overline{b_i(\boldsymbol{x},t)b_j(\boldsymbol{y},t)} = M_{ij}(\boldsymbol{r},\boldsymbol{R})$. Here we have defined the difference $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}$ and the mean $\boldsymbol{R} = (\boldsymbol{x}+\boldsymbol{y})/2$, keeping in mind that, for weakly inhomogeneous turbulence, the two-point correlation $M_{ij}(\boldsymbol{r},\boldsymbol{R})$ and, in fact, all two-point correlations below vary rapidly with $\boldsymbol{r}$ but vary slowly with $\boldsymbol{R}$. Taking the ensemble average of $h_{\rm G}$, we have

$$\overline{h}_{\rm G} = \frac{1}{4\pi}\int d^3R \int d^3r\epsilon_{ijk}M_{ij}(\boldsymbol{r},\boldsymbol{R})\frac{r_k}{r^3}. \tag{3}$$

Next, we suppose that the correlation scale l of the random small-scale field $\boldsymbol{b}$ is much smaller than the system scale R_S; i.e., we suppose that there exists an intermediate scale L such that $l \ll L \ll R_S$ with $M_{ij}(\boldsymbol{r},\boldsymbol{R}) \to 0$ as $|\boldsymbol{r}| \to L \gg l$. Then one can do the $\boldsymbol{r}$ integral even by restricting oneself to the intermediate scale L and still capture all the dominant contributions to the integral. This then motivates us to define the magnetic helicity density h of the random small-scale field as $\overline{h}_{\rm G} = \int d^3R\, h(\boldsymbol{R})$. Here,

$$h(\boldsymbol{R}) = \frac{1}{4\pi}\int_{L^3} d^3r\epsilon_{ijk}M_{ij}(\boldsymbol{r},\boldsymbol{R})\frac{r_k}{r^3}, \tag{4}$$

where we can formally let $L\to\infty$. The above expression for $h(\boldsymbol{R})$ in equation (4) is our proposal for the helicity density of the random small-scale field $\boldsymbol{b}$. Evidently, $h(\boldsymbol{R})$ is explicitly gauge-invariant. A qualitative description would be to say that the magnetic helicity density of a random small-scale field is the density of correlated links of the field. We can now derive the evolution equation for $h(\boldsymbol{R})$ and also meaningfully (in a gauge-invariant manner) talk about its flux. Note that this has not been possible before, although many papers (e.g., Blackman & Field 2000; Kleeorin et al. 2000; Vishniac & Cho 2001) have appealed to the notion of a magnetic helicity flux density in some qualitative fashion. We will see that the magnetic helicity evolution equation that we derive reproduces the known evolution equation for homogeneous turbulence and generalizes it to the inhomogeneous case by introducing possible fluxes of helicity.

3. MAGNETIC HELICITY DENSITY EVOLUTION

It is much simpler to work out the evolution equation for $h(\boldsymbol{R})$ by first going to Fourier space, using the two-scale approach of Roberts & Soward (1975), where all two-point correlations are assumed to vary rapidly with $\boldsymbol{r}$ and slowly with $\boldsymbol{R}$. Consider the equal time, ensemble average of the product $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)}$. The common dependence of f and g on t is assumed and will not explicitly be stated. Let $\hat{f}(\boldsymbol{k}_1)$ and $\hat{g}(\boldsymbol{k}_2)$ be the Fourier transforms of f and g, respectively. We can express this correlation as $\overline{f(\boldsymbol{x}_1)g(\boldsymbol{x}_2)} = \int \Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3k$, with

$$\Phi(\hat{f},\hat{g},\boldsymbol{k},\boldsymbol{R}) = \int \overline{\hat{f}(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{g}(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}e^{i\boldsymbol{K}\cdot\boldsymbol{R}}\, d^3K. \tag{5}$$

Here, $\boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_1-\boldsymbol{k}_2)$ and $\boldsymbol{K} = \boldsymbol{k}_1+\boldsymbol{k}_2$. We define in Fourier space the correlation and cross-correlation tensors of the $\boldsymbol{u}$- and $\boldsymbol{b}$-fields; $v_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_i,\hat{u}_j,\boldsymbol{k},\boldsymbol{R})$, $m_{ij}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{b}_i,\hat{b}_j,\boldsymbol{k},\boldsymbol{R})$, and $\chi_{jk}(\boldsymbol{k},\boldsymbol{R}) = \Phi(\hat{u}_j,\hat{b}_k,\boldsymbol{k},\boldsymbol{R})$. In MFD theory, the turbulent electromotive force (EMF) is given by $\overline{\mathcal{E}} = \overline{\boldsymbol{u}\times\boldsymbol{b}}$, whose components are $\overline{\mathcal{E}}_i(\boldsymbol{R}) = \epsilon_{ijk}\int\chi_{jk}(\boldsymbol{k},\boldsymbol{R})d^3k$. Furthermore, in Fourier space, we have for the magnetic helicity density

$$h(\boldsymbol{R}) = \int\int \epsilon_{ijk}\overline{\hat{b}_i(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_j(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}(ik_k/k^2)e^{i\boldsymbol{K}\cdot\boldsymbol{R}}\, d^3k\, d^3K. \tag{6}$$

We should remark that for an inhomogeneous system, the Coulomb gauge magnetic helicity density, say $\tilde{h} = \overline{\boldsymbol{a}\cdot\boldsymbol{b}}$, would have $(k_k+K_k/2)/(\boldsymbol{k}+\boldsymbol{K}/2)^2$ replacing k_k/k^2 in the Fourier space expression of equation (6). The two expressions are identical for the homogeneous case, and even in the weakly inhomogeneous case up to first-order terms in K/k, but not in general. So, $h(\boldsymbol{R}) \neq \tilde{h}(\boldsymbol{R})$ in general.

In order to compute the magnetic helicity evolution, we use the induction equation for $\boldsymbol{b}$ in Fourier space, $\partial\hat{b}_i(\boldsymbol{k})/\partial t = -\epsilon_{ipq}ik_p\hat{e}_q$. Here $\hat{e}_q$ is the Fourier transform of the small-scale electric field e_q, which is given by (e.g., Moffatt 1978)

$$\boldsymbol{e} = -\boldsymbol{u}\times\overline{\boldsymbol{B}} - \overline{\boldsymbol{U}}\times\boldsymbol{b} - \boldsymbol{u}\times\boldsymbol{b} + \overline{\boldsymbol{u}\times\boldsymbol{b}} + \eta\boldsymbol{j}. \tag{7}$$

Substituting this in the time derivative of equation (6), we get, after some straightforward algebra,

$$\begin{aligned}\frac{\partial h(\boldsymbol{R})}{\partial t} = \int\int\Big\{&-2\int\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_q(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\\ &+2(K_jk_q/k^2)\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_j(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\\ &-(K_sk_s/k^2)\overline{\hat{e}_q(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})\hat{b}_q(-\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{K})}\Big\}\\ &\times e^{i\boldsymbol{K}\cdot\boldsymbol{R}}\, d^3K\, d^3k.\end{aligned} \tag{8}$$

We denote the integrals over the three terms in curly brackets above as A_1, A_2, and A_3, respectively. From the definition of Φ in equation (5), the first term is simply $A_1 = -2\overline{\boldsymbol{e}\cdot\boldsymbol{b}}$, or

$$A_1 = 2\overline{\boldsymbol{b}\cdot(\boldsymbol{u}\times\boldsymbol{B})} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}} = -2\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}, \tag{9}$$

where $\overline{\mathcal{E}} = \overline{\boldsymbol{u}\times\boldsymbol{b}}$ is the turbulent EMF. Note that for homogeneous turbulence, only the term A_1 survives. This is because the other two terms, which involve a large K_i in the integrand, will introduce a large-scale R_i derivative when evaluating the integral over $\boldsymbol{K}$, and this vanishes in the homogeneous case. So, for the homogeneous case, we recover a local generalization (without volume integration) of the magnetic helicity conservation equation (Brandenburg & Subramanian 2005a):

$$\partial h/\partial t = -2\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}} \text{ (from } A_1 \text{ term only).} \quad (10)$$

In the inhomogeneous case, since the terms A_2 and A_3 are scalars that depend on K_i in the integrand, and hence a large-scale R_i derivative, they will contribute purely to the flux of helicity. The only term that involves volume generation of the helicity density is the A_1 term, which we see involves correlations no higher than the two-point one. This is in contrast to the current helicity evolution, which involved undetermined triple correlations in their volume generation (see SB04).

Let us now evaluate the helicity fluxes given by A_2 and A_3. This involves straightforward but tedious algebra. We also work out the flux to the lowest order in the R derivative. There are again three main types of contributions due to different parts of the electric field $\boldsymbol{e}$. First there is a contribution proportional to $\overline{\boldsymbol{B}}$ due to that part $\boldsymbol{e} = -\boldsymbol{u}\times\overline{\boldsymbol{B}} + \ldots$. In Fourier space, this gives $\hat{e}_q(\boldsymbol{k}) = \epsilon_{qlm}\int \hat{u}_m(\boldsymbol{k}-\boldsymbol{k}')\overline{B}_l(\boldsymbol{k}')dk'$. We substitute this into the expressions for A_2 and A_3, change the variables to $\boldsymbol{K}' = \boldsymbol{K} - \boldsymbol{k}'$, use the definition for χ_{ij}, and evaluate the integrations over $\boldsymbol{K}'$ and $\boldsymbol{k}'$, retaining only terms to lowest order in the R derivatives. We then get $A_2 = -\nabla_j\overline{\mathcal{F}}_j^{\rm VC}$ and $A_3 = -\nabla_j\overline{\mathcal{F}}_j^{A}$, where the mean field–dependent fluxes $\overline{\mathcal{F}}_j^{\rm VC}$ and $\overline{\mathcal{F}}_j^{A}$ are given by

$$\overline{\mathcal{F}}_i^{\rm VC} = \epsilon_{qlm}\overline{B}_l(\boldsymbol{R})\int ik_q\chi_{mi}k^{-2}\,d^3k,$$
$$\overline{\mathcal{F}}_i^{A} = -\epsilon_{qlm}\overline{B}_l(\boldsymbol{R})\int ik_i\chi_{mq}k^{-2}\,d^3k. \quad (11)$$

Note that $\overline{\mathcal{F}}_i^{A}$ only depends on the antisymmetric part of the cross-correlation χ_{mq}, whereas $\overline{\mathcal{F}}_i^{\rm VC}$ is sensitive to the symmetric part as well. Now consider the contribution proportional to the mean velocity from the part of the electric field $\boldsymbol{e} = -\overline{\boldsymbol{U}}\times\boldsymbol{b} + \ldots$. The evaluation of this follows the same steps as in evaluating equation (11), except that one can map $u_m \to b_l$ and $\overline{B}_l \to \overline{U}_m$. This gives $A_2 + A_3 = -\nabla_j\overline{\mathcal{F}}_j^{\rm bulk}$, where the flux due to bulk motions is given by

$$\overline{\mathcal{F}}_i^{\rm bulk} = \epsilon_{qlm}\overline{U}_m(\boldsymbol{R})\int(2ik_q m_{li} - ik_i m_{lq})k^{-2}\,d^3k. \quad (12)$$

Indeed, if the magnetic correlations were isotropic, then it is easy to simplify this further, and one gets $\overline{\mathcal{F}}_i^{\rm bulk} = hU_i$, exactly as one should for an advective flux!

The contribution to the fluxes from $\boldsymbol{e} = -\boldsymbol{u}\times\boldsymbol{b}$ (see eq. [7]) introduces triple correlations in the flux, which then need a closure theory to evaluate. We denote this flux term as $\overline{\mathcal{F}}_i^{\rm triple}$. However, since this triple correlation comes only in the flux, and not the volume generation term, it is likely that its value can be constrained by a conservation law. This will be examined in more detail in the future. Note also that the contribution to the helicity flux from $\boldsymbol{e} = \overline{\boldsymbol{u}\times\boldsymbol{b}} + \ldots$ is zero and that the resistive contribution from $\boldsymbol{e} = \eta\boldsymbol{j} + \ldots$ is likely to be negligible compared to the terms that we retain.

Putting all our results together, we can write for the evolution of the magnetic helicity density

$$\partial h/\partial t + \boldsymbol{\nabla}\cdot\overline{\mathcal{F}} = -2\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}} - 2\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}, \quad (13)$$

where the flux is $\overline{\mathcal{F}}_i = \overline{\mathcal{F}}_i^{\rm VC} + \overline{\mathcal{F}}_i^{A} + \overline{\mathcal{F}}_i^{\rm bulk} + \overline{\mathcal{F}}_i^{\rm triple}$. We should emphasize that equation (13) is a *local* magnetic helicity conservation law. If one were unable to define a gauge-invariant magnetic helicity density, one would only have an integral (global) conservation law, as in previous studies. Further simplification of the helicity fluxes for use in, say, MFD models requires the evaluation of the turbulent EMF tensor χ_{ij}, which can be done only under a closure scheme and will be presented elsewhere. It turns out that $\overline{\mathcal{F}}_i^{\rm VC}$ is a generalization of the magnetic helicity flux obtained by Vishniac & Cho (2001), which is particularly important in the presence of strong shear (Brandenburg & Subramanian 2005b). The fluxes used by Kleeorin et al. (2000) can arise from both $\overline{\mathcal{F}}_i^{A}$ and the contributions of the antisymmetric parts of the correlations to $\overline{\mathcal{F}}_i^{\rm VC}$. The magnetic helicity fluxes also vanish if the turbulence is homogeneous. For isotropic but inhomogeneous turbulence, $\overline{\mathcal{F}}_i^{\rm VC}$ also depends purely on the antisymmetric part of χ_{ij}, just like $\overline{\mathcal{F}}_i^{A}$. Furthermore, the VC and A fluxes are proportional to two-point correlations (χ_{ij}) and the mean field $\overline{\boldsymbol{B}}$ (see eq. [11]); this is unlike the two-dimensional case (Silvers 2006).

Since the small-scale magnetic helicity opposes the kinetic part of the α-effect (Pouquet et al. 1976), its loss through corresponding magnetic helicity fluxes can alleviate this quenching effect (Blackman & Field 2000; Kleeorin et al. 2000; Vishniac & Cho 2001). Note that the α-effect quantifies the contribution of $\overline{\mathcal{E}}$ that is aligned with the mean field; i.e., $\overline{\mathcal{E}} = \alpha\overline{\boldsymbol{B}} + \ldots$ for the simplest case of a scalar α-effect. For a closed system, equation (10) applies, and, in the stationary limit, this predicts $\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}} = -\eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$, which tends to zero as $\eta\to 0$ for any reasonable spectrum of current helicity. This leads to a catastrophic quenching of the turbulent EMF parallel to $\overline{\boldsymbol{B}}$. In the presence of helicity fluxes, however, we have $\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}} = -\frac{1}{2}\boldsymbol{\nabla}\cdot\overline{\mathcal{F}} - \eta\overline{\boldsymbol{j}\cdot\boldsymbol{b}}$ in the stationary limit, and so $\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}}$ need not be catastrophically quenched. The turbulent magnetic helicity fluxes worked out here are therefore crucial for the efficient working of the mean field dynamo. Numerical work in determining the α-effect did show a 30-fold increase in simulations that allowed helicity fluxes to develop (Brandenburg & Sandin 2004). However, a more convincing demonstration of the importance of helicity fluxes comes from a dynamo simulation in the presence of shear showing that only with open boundaries can a significant large-scale field of equipartition field strength develop (Brandenburg 2005).

4. CONCLUSIONS

We have proposed here a local gauge-invariant definition of magnetic helicity density for random fields in weakly inhomogeneous systems, which can also have boundaries. This is particularly useful in the context of MFDs since one can then meaningfully discuss magnetic helicity fluxes and the local effect of Lorentz forces. We have derived an evolution equation for the local magnetic helicity density and showed that they naturally involve helicity fluxes, which may alleviate the problems associated with MFDs (Shukurov et al. 2006). Our work therefore lays the conceptual foundation for the many discussions of the effects of helicity fluxes already existing in the literature and for future explorations.

Future applications might include the use of equation (4) together with an assumption of isotropy to estimate the spatial variation of magnetic helicity density by measuring at least one of the off-diagonal components of $M_{ij}(\boldsymbol{r}, \boldsymbol{R})$. This type of approach has been adopted by Matthaeus et al. (1982) to determine the magnetic helicity in the solar wind by measuring just a time series of an off-diagonal component of $M_{ij}(\boldsymbol{r})$ under the Taylor hypothesis. However, no dependence on the large-scale coordinate $\boldsymbol{R}$ has been determined. In principle, similar ideas could be applied to determine $h(\boldsymbol{R})$ on the solar surface without necessarily having access to the dependence of the magnetic field with depth.

We thank the organizers and participants of the program "Magnetohydrodynamics of Stellar Interiors" at the Isaac Newton Institute in Cambridge (UK) for a stimulating environment that led to the present work. K. S. thanks NORDITA for their hospitality during the course of this work.

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