Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

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1. Computing source function from observed center to limb variation. We use the equation of radiation transport to infer the depth dependence of the source function from the observed center to limb variation.

We use the gray approximation and write the equation of radiation transport in the form

$$\hat{\mathbf{n}} \cdot \nabla I = -\kappa\rho(I - S)$$

where $I(z, \hat{\mathbf{n}})$ is the intensity as a function of height z and direction $\hat{\mathbf{n}}$, κ is the opacity per unit mass, ρ is the density of the gas, and S is the source function.

(a) Derive the equation of radiation transport in the form

$$\mu \frac{dI}{d\tau} = I - S \quad (1)$$

where $\mu = \cos\theta$ is the cosine of the angle θ between the direction of the ray $\hat{\mathbf{n}}$ and the vertical direction, and τ is the optical depth. Thus, I is now a function of τ and μ.

(b) Verify that

$$I(\tau, \mu) = I(\tau_0, \mu)\, e^{-(\tau_0 - \tau)/\mu} + \int_\tau^{\tau_0} S(\tau')\, e^{-(\tau' - \tau)/\mu}\, d\tau'/\mu$$

obeys Eq. (1), where $I(\tau_0, \mu)$ is the intensity at some arbitrarily chosen reference value of the optical depth τ_0.

(c) Put $\tau = 0$ and $\tau_0 = \infty$, and show that

$$I(\mu) = \int_0^\infty S(\tau')\, e^{-\tau'/\mu}\, d\tau'/\mu$$

where $I(\mu)$ is the intensity at the position of the observer, who is located at $\tau = 0$, so $I(\mu)$ is the same as $I(0, \mu)$. This equation is an integral equation that can be inverted to obtain $S(\tau)$ for a given profile of $I(\mu)$.

(d) Drop the prime in the equation above and show that

$$\int_0^\infty \tau\, e^{-\tau/\mu}\, d\tau/\mu = \mu^2$$

and

$$\int_0^\infty \tau^2\, e^{-\tau/\mu}\, d\tau/\mu = 2\mu^3$$

(e) Insert for $S(\tau)$ the Taylor expansion

$$S(\tau) = S_0 + S_1\tau + \tfrac{1}{2}S_2\tau^2$$

where S_0, S_1, and S_2, are suitable coefficients and compute $I(\mu)$.

(f) Compute $I(\mu)$ for

$$S(\tau) = \frac{2}{5} + \frac{3}{5}\tau. \quad (2)$$

(g) Use the measured and tabulated values of $I(\mu)$ to compute $S(\tau)$.

μ	$I(\mu)$
1	1.0
2/3	0.8
1/3	0.5

(h) Describe how this new function $S(\tau)$ obtained from the tabulated values is different from that given by Eq. (2). Plot (or sketch) both $I(\mu)$ and $S(\tau)$.

(i) Instead of performing an expansion around $\tau = 0$ as under (e), expand around $\tau = \tau_* = 1$, using

$$S(\tau) = S_0 + S_1(\tau - \tau_*) + \tfrac{1}{2} S_2 (\tau - \tau_*)^2, \tag{3}$$

with new coefficients S_0, S_1, and S_2, and repeat steps (f)–(h).

. . . .

(a) In spherical coordinates, the unit vector $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

However, since $I = I(z, \hat{\mathbf{n}})$, ∇I has only a z component, i.e.,

$$\hat{\mathbf{n}} \cdot \nabla I = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \partial/\partial z \end{pmatrix} = \cos\theta \frac{\partial I}{\partial z} = \cos\theta \frac{dI}{dz},$$

where we have used the fact that I depends only on the z coordinate, and there is no differentiation with respect to μ. Since $\cos\theta = \mu$ we have

$$\mu \frac{dI}{dz} = -\kappa\rho(I - S)$$

We can now use the definition of optial depth, $d\tau = -\kappa\rho dz$ to obtain

$$\mu \frac{dI}{d\tau} = I - S.$$

(b) We verify this equation by inserting $dI/d\tau$, so we differentiate the given expression:

$$\frac{dI}{d\tau} = \frac{1}{\mu} I(\tau_0, \mu) e^{-(\tau_0 - \tau)/\mu} - \frac{1}{\mu} S(\tau) e^{-(\tau - \tau)/\mu} + \int_\tau^{\tau_0} S(\tau') e^{-(\tau' - \tau)/\mu} d\tau'/\mu^2$$

where the differentiation of the integral consisted of two parts: "undoing" the integration (noting the variable integration boundary is here the lower one) and the differentiation of the integrand. Multiplying by μ yields

$$\mu \frac{dI}{d\tau} = I(\tau_0, \mu) e^{-(\tau_0 - \tau)/\mu} - S(\tau) + \int_\tau^{\tau_0} S(\tau') e^{-(\tau' - \tau)/\mu} d\tau'/\mu$$

where sum of the first and the last term on the rhs equals I, and so we have

$$\mu \frac{dI}{d\tau} = I - S$$

(c) For $\tau_0 = \infty$, the first exponential factor vanishes, i.e., $e^{-(\tau_0-\tau)/\mu} = 0$ and so, putting also $\tau = 0$ yields directly

$$I(0,\mu) = \int_0^\infty S(\tau')\, e^{-\tau'/\mu}\, d\tau'/\mu$$

The rest is just renaming $I(0,\mu)$ into just $I(\mu)$.

(d) Let us begin with

$$\int_0^\infty e^{-\tau/\mu}\, d\tau = -\,\mu e^{-\tau/\mu}\Big|_0^\infty = \mu$$

Next, using integration by parts, we have

$$\int_0^\infty \underbrace{\tau}_{u}\,\underbrace{e^{-\tau/\mu}}_{v'}\, d\tau = \underbrace{\tau}_{u}\,\underbrace{(-\mu)e^{-\tau/\mu}}_{v}\,|_0^\infty - \int_0^\infty \underbrace{1}_{u'}\,\underbrace{(-\mu)e^{-\tau/\mu}}_{v}\, d\tau$$

The first term vanishes on both boundaries, and the second term gives μ times the previous integral, which was also μ, so we obtain μ^2,

$$\int_0^\infty \underbrace{\tau}_{u}\,\underbrace{e^{-\tau/\mu}}_{v'}\, d\tau = \mu^2.$$

Finally,

$$\int_0^\infty \underbrace{\tau^2}_{u}\,\underbrace{e^{-\tau/\mu}}_{v'}\, d\tau = \underbrace{\tau^2}_{u}\,\underbrace{(-\mu)e^{-\tau/\mu}}_{v}\,|_0^\infty - \int_0^\infty \underbrace{2\tau}_{u'}\,\underbrace{(-\mu)e^{-\tau/\mu}}_{v}\, d\tau$$

Again, the first term vanishes on both boundaries, and the second term gives μ times the previous integral, which was μ^2, so we obtain $2\mu^3$,

(e) Inserting the integrals above into the expression in (c), we have

$$I(\mu) = S_0 + S_1\mu + S_2\mu^2$$

(f) We can just use (f) and see that $S_0 = 2/5$, $S_1 = 3/5$, and $S_2 = 0$, so

$$I(\mu) = \frac{2}{5} + \frac{3}{5}\mu.$$

(g) We have 3 measurements, each one resulting in one equation with 3 unknowns,

$$I(\mu) = S_0 + S_1\mu + S_2\mu^2$$

so for the 3 data values $I(\mu_1)$, $I(\mu_2)$, and $I(\mu_3)$ with $\mu_1 = 1$, $\mu_2 = 2/3$, and $\mu_3 = 1/3$, we have 3 equations that we arrange immediately in matrix form, i.e.,

$$\begin{pmatrix} I(\mu_1) \\ I(\mu_2) \\ I(\mu_3) \end{pmatrix} = \begin{pmatrix} 1 & \mu_1 & \mu_1^2 \\ 1 & \mu_2 & \mu_2^2 \\ 1 & \mu_3 & \mu_3^2 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix}.$$

In our case with $\mu_1 = 1$, $\mu_2 = 2/3$, and $\mu_3 = 1/3$, the matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2/3 & 4/9 \\ 1 & 1/3 & 1/9 \end{pmatrix}$$

which we invert,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2/3 & 4/9 \\ 1 & 1/3 & 1/9 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ -9/2 & 12 & -15/2 \\ 9/2 & -9 & 9/2 \end{pmatrix}$$

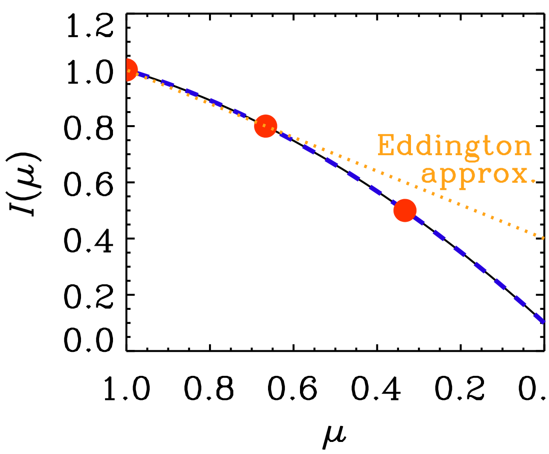
so the solution for the 3 unknowns is

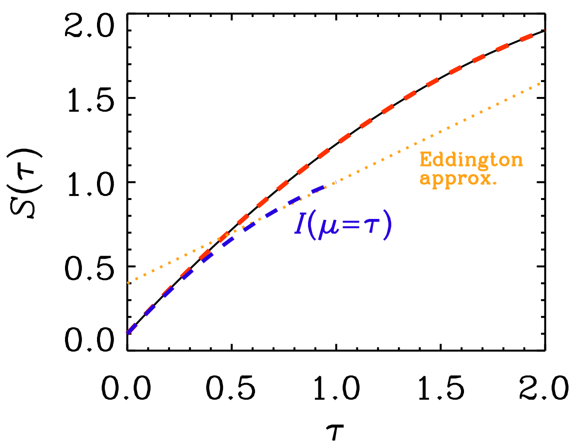
$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 \\ -9/2 & 12 & -15/2 \\ 9/2 & -9 & 9/2 \end{pmatrix} \begin{pmatrix} I(\mu_1) \\ I(\mu_2) \\ I(\mu_3) \end{pmatrix}$$

Inserting now our measurements, we obtain

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 \\ -9/2 & 12 & -15/2 \\ 9/2 & -9 & 9/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0.8 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1/10 \\ 27/20 \\ -9/20 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 1.35 \\ -0.45 \end{pmatrix}$$

(h) The first plot below shows the original data points together with $I(\mu) = S_0 + S_1\mu + S_2\mu^2$. Note that $\mu = 1$ (disk center) is here to the left, and $\mu = 0$ (limb) is to the right. The second plot shows $S(\tau) = S_0 + S_1\tau + \frac{1}{2}S_2\tau^2$.





The solution given by Eq. (2) matches the new one at $\tau = 1/2$.

(i) We start with

$$S(\tau) = S_0 + S_1(\tau - \tau_*) + \tfrac{1}{2}S_2(\tau - \tau_*)^2, \tag{4}$$

$$S(\tau) = S_0 + S_1\tau - S_1\tau_* + \tfrac{1}{2}S_2(\tau^2 - 2\tau\tau_* + \tau_*^2)^2, \tag{5}$$

and rewrite

$$S(\tau) = \underbrace{[S_0 - S_1\tau_* + \tfrac{1}{2}S_2\tau_*^2]}_{\tilde{S}_0} + \underbrace{[S_1 - S_2\tau_*]}_{\tilde{S}_1}\tau + \tfrac{1}{2}S_2\tau^2, \tag{6}$$

and $\tilde{S}_2 = S_2$. Again, we have 3 equations that we can write in matrix form, so for $\tau_* = 1$, we have

$$\begin{pmatrix} \tilde{S}_0 \\ \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix}.$$

The inverse is given by

$$\begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The coefficients turn out to be $\tilde{S}_0 = 1.225$, $\tilde{S}_1 = 0.9$, and $\tilde{S}_2 = -0.45$, but the functional form of $S(\tau)$ is the same as for $\tau_* = 0$.

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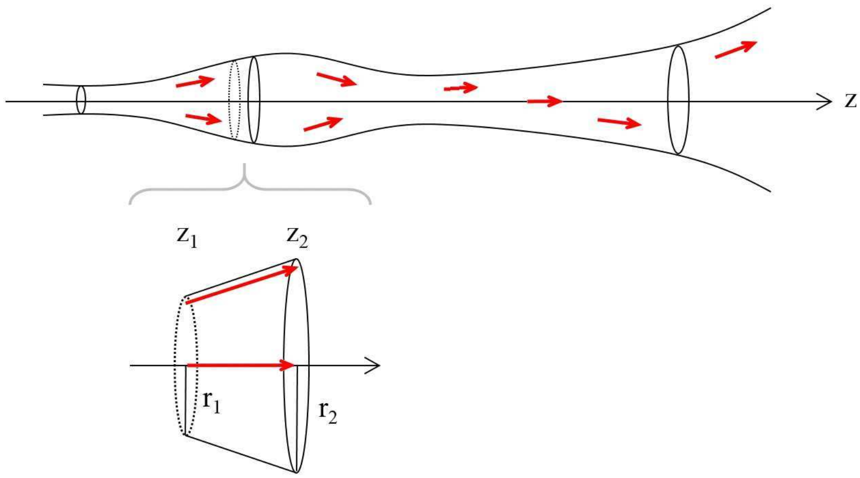
2. Thin magnetic flux tubes. We'd like to explore the real meaning of the most trivial-looking of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$.

Consider a bundle of magnetic field lines constrained to follow a sausage-like shape oriented mostly along the z axis:



We'll use cylindrical coordinates (r, ϕ, z) to describe spatial variations in $\mathbf{B}$, and assume axial symmetry around the z axis (i.e., no variations in ϕ, and $B_\phi = 0$).

(a) In the zoomed-in region between z_1 and z_2, assume that

$$\frac{\partial B_z}{\partial z} \quad \text{is a constant given by} \quad \frac{\Delta B_z}{\Delta z} = \frac{B_{z2} - B_{z1}}{z_2 - z_1} \ .$$

Integrate the $\nabla \cdot \mathbf{B} = 0$ equation to show that

$$B_r = C r$$

between z_1 and z_2, and solve for C in terms of the other variables of this problem. Describe any assumptions or boundary conditions that you had to use.

(b) Note that, along the outer edge of the tube, the vector $\mathbf{B}$ points along the direction of the tube's geometric outline. Considering only *small* relative changes in r between z_1 and z_2, use everything given so far to show that the change in cross-sectional area ($A = \pi r^2$) from z_1 to z_2 is given by

$$\frac{\Delta A}{A_1} \approx -\frac{\Delta B_z}{B_{z1}} \ .$$

Hint: The binomial expansion for the quantity $[1 + (\Delta r / r_1)]^2$, where $\Delta r = (r_2 - r_1) \ll r_1$, might be useful to use at some point.

(c) Lastly, assume that all "Δ" changes are infinitesimally small, in comparison to the values of the quantities at z_1, and show that

$$A(z) B_z(z) = \text{ constant}$$

i.e., that *magnetic flux is conserved* along a thin flux tube.

. . . .

(a) In cylindrical coordinates with symmetry properties as described above, $\nabla \cdot \mathbf{B} = 0$ is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

and if $\partial B_z/\partial z$ is a constant, then this reduces to an ordinary differential equation, which we can write as

$$\frac{dy}{dr} = -r\left(\frac{\Delta B_z}{\Delta z}\right) \quad \text{where} \quad y = rB_r \; .$$

This equation is separable and can be integrated,

$$\int dy = -\left(\frac{\Delta B_z}{\Delta z}\right)\int r\,dr$$

$$y = rB_r = -\left(\frac{\Delta B_z}{\Delta z}\right)\left[\frac{r^2}{2} + \tilde{c}\right]$$

Dividing both sides by r gives one term that's proportional to r (which is what the problem is asking for) plus another term that depends on the integration constant $\tilde{c}$. We can argue that $\tilde{c}$ must be set to zero, since it's clear from the cartoon that B_r should go to zero along the axis of the flux tube ($r = 0$). Thus,

$$B_r = -\frac{r}{2}\frac{\Delta B_z}{\Delta z} \quad \text{i.e.,} \quad C = -\frac{1}{2}\frac{\Delta B_z}{\Delta z} \; .$$

(b) If the magnetic field between z_1 and z_2 is oriented like it is in the figure, then the magnetic field vector is parallel to the "displacement vector" that points from coordinates (r_1,z_1) to (r_2,z_2). In other words, the components are proportional to one another as follows:

$$\frac{B_{r1}}{B_{z1}} = \frac{\Delta r}{\Delta z}$$

where $\Delta r = (r_2 - r_1)$ and $\Delta z = (z_2 - z_1)$. We'll use this in a bit. Next, we can begin writing the change in cross-sectional area as

$$\Delta A = A_2 - A_1 = (\pi r_2^2) - (\pi r_1^2) = \pi r_1^2\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] = \pi r_1^2\left[\left(\frac{r_1 + \Delta r}{r_1}\right)^2 - 1\right]$$

or, in other words,

$$\frac{\Delta A}{A_1} = \left(1 + \frac{\Delta r}{r_1}\right)^2 - 1 \; .$$

For small relative changes ($\Delta r \ll r_1$), the binomial expansion gives

$$\left(1 + \frac{\Delta r}{r_1}\right)^2 \approx 1 + 2\frac{\Delta r}{r_1} + \{\text{smaller terms}\}$$

and

$$\frac{\Delta A}{A_1} \approx 2\frac{\Delta r}{r_1} = \frac{2}{r_1}\left(\frac{B_{r1}\,\Delta z}{B_{z1}}\right) \qquad \text{(from the above proportionality relation)} \; .$$

But, from part (a), we know that

$$B_{r1} = -\frac{r_1}{2}\frac{\Delta B_z}{\Delta z}$$

and plugging that in gives us the desired answer,

$$\frac{\Delta A}{A_1} \approx -\frac{\Delta B_z}{B_{z1}} .$$

(c) If the "deltas" are infinitesimally small, they can be changed to derivatives; i.e.,

$$\frac{dA}{A} = -\frac{dB_z}{B_z} .$$

Since we're talking about changes mainly along the z axis, the above is equivalent to

$$\frac{1}{A}\frac{dA}{dz} = -\frac{1}{B_z}\frac{dB_z}{dz} .$$

We can show this is exactly what one would get if AB_z = constant, since that would be the same as saying

$$\begin{aligned} \frac{d}{dz}\left[A(z)\,B_z(z)\right] &= 0 \\ A\frac{dB_z}{dz} + B_z\frac{dA}{dz} &= 0 \end{aligned}$$

and if we divide each term on the left-hand side by AB_z, we get the boxed equation above.

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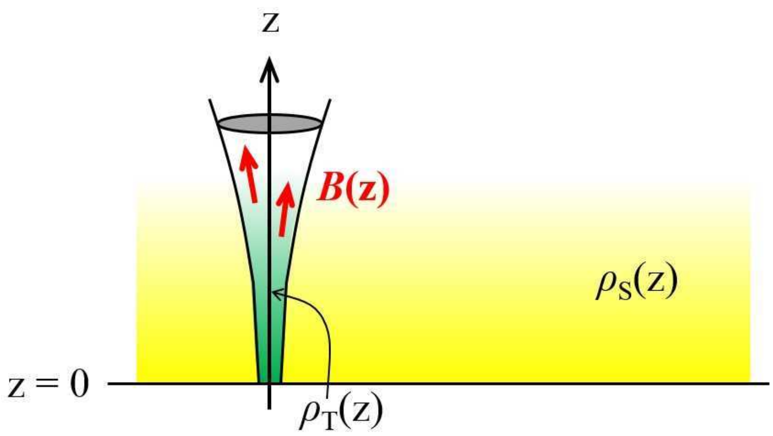
3. Magnetic flux tubes in the Sun's atmosphere. Consider a small piece of the solar surface, in which thin flux tubes with strong **B** are peppered through larger regions of weak/negligible **B**:



Let's assume the entire region shown above is isothermal (i.e., constant T throughout), and the z dependence of density ρ obeys the hydrostatic equilibrium derived in class,

$$\rho(z) = \rho_0 \exp\left(-\frac{z}{H}\right) \quad \text{where} \quad H = \frac{k_B T}{\mu m_H g} .$$

where the density at the base ($z = 0$) inside the tube $\rho_{0,T}$ does not necessarily equal the base density in the surroundings $\rho_{0,S}$.

(a) If the plasma inside the tube is in total pressure balance with the plasma outside the tube, show that the z dependence of magnetic field strength B inside the tube obeys

$$B(z) = B_0 \exp\left(-\frac{z}{K}\right)$$

and solve for B_0 and K in terms of the other properties of the system.

(b) Using the principle of magnetic flux conservation from the previous problem, note that if $B(z)$ decreases with increasing height, then the cross-sectional area of the tube $A(z)$ must increase. If the tubes occupy 1% of the solar surface at the lower boundary ($z = 0$), then at what height will they fill the entire surface? Solve for this "merging height" both in terms of the other variables and also as an actual number (in units of km) for the real solar case of $T \approx 5000$ K and $\mu \approx 1.3$.

. . . .

(a) The total pressure is given by

$$P_{\rm tot} = P_{\rm gas} + P_{\rm mag} = \frac{\rho k_B T}{\mu m_H} + \frac{B^2}{2\mu_0} .$$

The interior of the tube has both gas and magnetic pressure. The surroundings have weak or negligible magnetic field, so that allows us to assume that the surroundings have *only* gas pressure. Thus, if the total pressure is equal at a given height z in the two regions, then

$$\begin{aligned} P_{\rm tot,T} &= P_{\rm tot,S} \\ \frac{\rho_T k_B T}{\mu m_H} + \frac{B^2}{2\mu_0} &= \frac{\rho_S k_B T}{\mu m_H} \end{aligned}$$

and thus, we can solve for

$$\frac{B^2}{2\mu_0} = \frac{(\rho_S - \rho_T) k_B T}{\mu m_H} = \left[\frac{(\rho_{0,S} - \rho_{0,T}) k_B T}{\mu m_H}\right] \exp\left(-\frac{z}{H}\right) .$$

Thus, we can solve for

$$B(z) = B_0 \exp\left(-\frac{z}{K}\right)$$

where

$$B_0 = \sqrt{\frac{2\mu_0(\rho_{0,S} - \rho_{0,T}) k_B T}{\mu m_H}} \qquad K = 2H .$$

(b) When magnetic flux is conserved, the last problem told us that the product $B(z)A(z)$ should remain constant. Thus,

$$B_0 A_0 = B_m A_m$$

where subscript '0' refers to the photospheric lower boundary, and subscript 'm' refers to the merging height at which the flux tubes expand to fill the volume.

We don't need absolute values for the areas A_0 and A_m. We know that A_0 is 1% of the solar surface, and A_m is essentially 100% of the solar surface, so we can write the above formula as a *ratio*...

$$\frac{B_m}{B_0} = \frac{A_0}{A_m} = 0.01 .$$

We also know from part (a) that the magnetic field strength B_m at some height z_m above the photosphere can be written as

$$B_m = B_0 \exp\left(-\frac{z_m}{2H}\right) \quad \text{i.e.,} \quad \frac{B_m}{B_0} = 0.01 = \exp\left(-\frac{z_m}{2H}\right)$$

which can be solved for

$$z_m = -2H \ln(0.01)$$

which must be a positive number, since $\ln(0.01) = -4.605$. Plugging in the numbers given, the scale height $H = 116$ km. Thus,

$$z_m = 1067 \text{ km}$$

which is in the middle of the chromosphere.
