

1. In Lecture 5, it was shown that to leading order the center-to-limb variation is given by

$$I_\nu(\mu) = B_\nu - \frac{\cos \theta}{\rho \kappa_\nu} \frac{dB_\nu}{dr} \quad (1)$$

Compare this solution with that found for problem 1 of Homework 2 assuming that the source function is $S = B_\nu$ with $S(\tau) = 0.1 + 1.35\tau - \frac{1}{2}0.45\tau^2$. Sketch this solution in the key to Homework 2 for problem 1.

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 We drop the subscript ν (gray approximation) and write the r derivative in terms of a τ derivative by using

$$-\rho \kappa dr = d\tau.$$

Furthermore, $\cos \theta = \mu$, so Eq. (1) takes the form

$$I_\nu(\mu, \tau) = S(\tau) + \mu \frac{dS(\tau)}{d\tau}$$

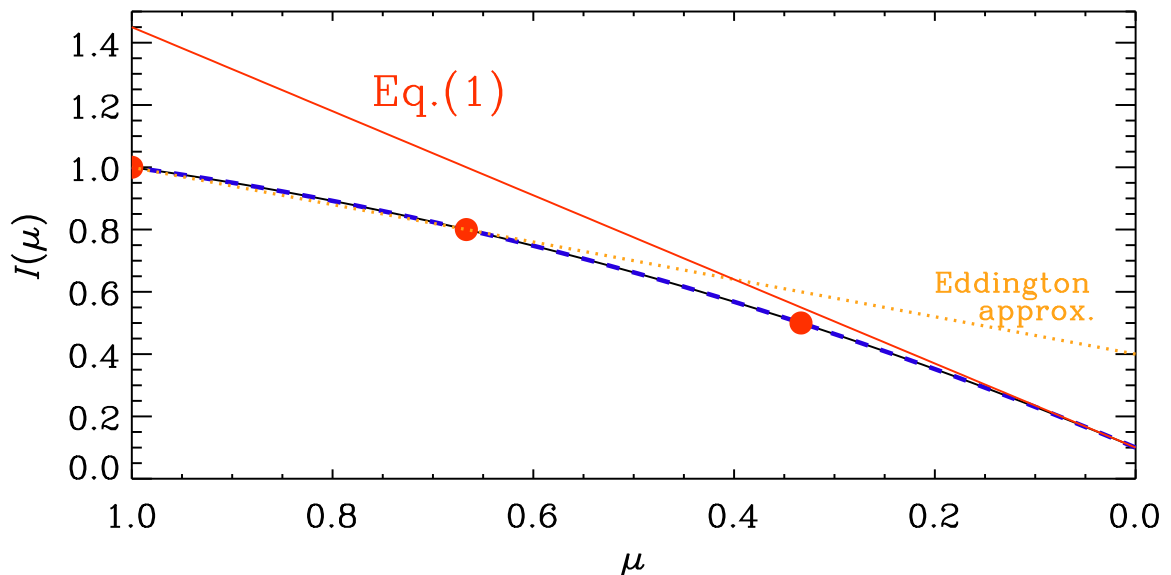
Inserting now the solution $S(\tau) = 0.1 + 1.35\tau - \frac{1}{2}0.45\tau^2$, we find

$$I_\nu(\mu, \tau) = 0.1 + 1.35\tau - 0.45\tau^2 + \mu(1.35 - 0.9\tau)$$

Applying this to $\tau = 0$ (because the observer is at $\tau \rightarrow 0$), we have

$$I_\nu(\mu, \tau = 0) = 0.1 + 1.35\mu.$$

This solution is overplotted in the figure below and labelled as “Eq.(1)”. It corresponds to the tangent of the exact solution through $\mu = 0$.



2. In Lecture 4, we looked at the periodic table and computed the energy mc^2 gain in the reaction $4\text{H} \rightarrow \text{He}$. Compare with the corresponding energy gain (or loss) for the hypothetical reactions $7\text{H} \rightarrow \text{Li}$, $9\text{H} \rightarrow \text{Be}$, $10\text{H} \rightarrow \text{B}$, $12\text{H} \rightarrow \text{C}$, $14\text{H} \rightarrow \text{N}$, $16\text{H} \rightarrow \text{O}$, $19\text{H} \rightarrow \text{F}$, and $20\text{H} \rightarrow \text{Ne}$.

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 We use $m_{\text{H}} = 1.008$ and denote by n the number of nucleons of the target product, whose mass is denoted by m_{target} . The last column gives the mass difference per nucleon, i.e., $(nm_{\text{H}} - m_{\text{target}})/n$.

reaction	nm_{H}	m_{target}	$nm_{\text{H}} - m_{\text{target}}$	per nucleon
$4\text{H} \rightarrow \text{He}$	4.032	4.0026	0.0294	0.0073
$7\text{H} \rightarrow \text{Li}$	7.056	6.94	0.116	0.0166
$9\text{H} \rightarrow \text{Be}$	9.072	9.012	0.060	0.0067
$10\text{H} \rightarrow \text{B}$	10.08	10.81	-0.73	-0.073
$12\text{H} \rightarrow \text{C}$	12.096	12.011	0.085	0.0071
$14\text{H} \rightarrow \text{N}$	14.112	14.007	0.105	0.0075
$16\text{H} \rightarrow \text{O}$	16.128	15.999	0.129	0.0081
$19\text{H} \rightarrow \text{F}$	19.152	18.9984	0.1536	0.0081
$20\text{H} \rightarrow \text{Ne}$	20.16	20.1797	-0.0197	-0.0010

The energy gain per nucleon would be the largest for lithium. The problem here is that lithium is destroyed for temperatures slightly above 2 million K.

3. Using the Planck law,

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_{\text{B}}T} - 1}, \quad (2)$$

to derive the limits for large and small frequencies ν .

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 For small frequencies, $\nu \rightarrow 0$, we have $e^{h\nu/k_{\text{B}}T} \approx 1 + h\nu/k_{\text{B}}T$, so we have

$$I(\nu, T) \approx \frac{2h\nu^3}{c^2} \frac{1}{1 + h\nu/k_{\text{B}}T - 1} = \frac{2h\nu^3}{c^2} \frac{1}{h\nu/k_{\text{B}}T} = \frac{2\nu^2}{c^2} k_{\text{B}}T.$$

For large frequencies, $\nu \rightarrow \infty$, the exponential function dominates compared with unity, so we can write

$$I(\nu, T) = \frac{2h\nu^3}{c^2} e^{-h\nu/k_{\text{B}}T}.$$

4. Ignoring effects of the atmosphere, what is the theoretical effective temperature on Titan, which has a distance of 1.427×10^{12} m from the Sun. You may use the material of Lecture 4 for this exercise.

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 In lecture 4, we derived the formula

$$T_{\text{eff}}^{\text{Earth}} = T_{\text{eff}}^{\text{Sun}} \left(\frac{R_{\text{Sun}}}{2D} \right)^{1/2}$$

Applying this to Titan, we have

$$T_{\text{eff}}^{\text{Titan}} = 5778 \times \left(\frac{7 \times 10^8}{2 \times 1.427 \times 10^{12}} \right)^{1/2} \text{ K} = 90 \text{ K}$$

This ignores the effects of albedo and greenhouse effects. Interestingly, the greenhouse effect is a negative one for Titan. If one takes Titan's albedo of $A = 0.21$ into account, one has an extra $(1 - A)^{1/4} \approx 0.94$ factor, so $T_{\text{eff}}^{\text{Titan}} = 85 \text{ K}$.

5. Using index notation, show that

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} \tag{3}$$

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 We insert the formulae for cross products and compute the i th component of $\nabla \times \nabla \times \mathbf{A}$ as follows:

$$(\nabla \times \nabla \times \mathbf{A})_i = \epsilon_{ijk} \partial_j (\nabla \times \mathbf{A})_k = \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l A_m = \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m.$$

Next, we use the formula for double epsilons: $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$, and obtain

$$(\nabla \times \nabla \times \mathbf{A})_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l A_m = \partial_j \partial_i A_j - \partial_j \partial_j A_i.$$

Here, $\partial_j \partial_i A_j = \partial_i (\partial_j A_j)$ (because ∂_j and ∂_i commute), but since $\partial_j A_j$ is just the divergence of \mathbf{A} , and the unbalanced ∂_i corresponds to a gradient, we have $\nabla(\nabla \cdot \mathbf{A}) = \nabla \nabla \cdot \mathbf{A}$. The operator $\partial_j \partial_j$ is just the Laplacian, ∇^2 , so we have

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$$

6. (a) Using the material of Lecture 8, show that $Q^2 + U^2 + V^2 = I^2$. (b) Discuss why this relation is not obeyed for "wave packets" of finite length (see Sect. 3.5.2 of Stix 2002).

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 The x and y components of the electric field vector are given by $E_x = \xi_x \cos \phi$ and $E_y = \xi_y \cos(\phi + \epsilon)$. The corresponding 4 Stokes parameters are then

$$I = \xi_x^2 + \xi_y^2, \quad Q = \xi_x^2 - \xi_y^2, \quad U = 2\xi_x \xi_y \cos \epsilon, \quad V = 2\xi_x \xi_y \sin \epsilon.$$

Next, we compute $Q^2 + U^2 + V^2$, i.e.,

$$Q^2 + U^2 + V^2 = (\xi_x^2 - \xi_y^2)^2 + (2\xi_x \xi_y \cos \epsilon)^2 + (2\xi_x \xi_y \sin \epsilon)^2 = (\xi_x^2 - \xi_y^2)^2 + 4\xi_x^2 \xi_y^2 (\cos^2 \epsilon + \sin^2 \epsilon)$$

Using $\cos^2 \epsilon + \sin^2 \epsilon = 1$, we have

$$Q^2 + U^2 + V^2 = (\xi_x^2 - \xi_y^2)^2 + 4\xi_x^2 \xi_y^2 = \xi_x^4 - 2\xi_x^2 \xi_y^2 + \xi_y^4 + 4\xi_x^2 \xi_y^2 = \xi_x^4 + 2\xi_x^2 \xi_y^2 + \xi_y^4$$

and so

$$Q^2 + U^2 + V^2 = \xi_x^4 + 2\xi_x^2 \xi_y^2 + \xi_y^4 = (\xi_x^2 + \xi_y^2)^2 = I^2.$$

For wave packets of finite length, we have to take averages, so

$$U^2 = 4\langle \xi_x \xi_y \cos \epsilon \rangle^2 \leq 4\langle \xi_x \xi_y \rangle^2 \langle \cos^2 \epsilon \rangle, \quad V^2 = 4\langle \xi_x \xi_y \sin \epsilon \rangle^2 \leq 4\langle \xi_x \xi_y \rangle^2 \langle \sin^2 \epsilon \rangle,$$

and therefore $Q^2 + U^2 + V^2 \leq I^2$.

7. What is the mean molecular weight μ for an ionized mixture with hydrogen abundance of (i) $X = 0.735$ or (ii) $X = 0.739$, assuming $Y = 1 - X$ for the helium abundance.

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In lecture 10, we derived the formula

$$\mu = \frac{1}{X + Y/4}$$

for a *neutral* gas. This is because the fractional number densities of H and He are respectively

$$n_{\text{H}} = \frac{\rho X}{m_{\text{H}}}, \quad n_{\text{He}} = \frac{\rho Y}{m_{\text{He}}}.$$

In the (singly) ionized case, there are twice as many particles, i.e.,

$$\mu = \frac{1}{2X + 2Y/4} = 0.624 \quad \text{or} \quad 0.599$$

for cases (i) and (ii), respectively. However, helium can also be doubly ionized, in which case we have 3 particles per helium atom, so

$$\mu = \frac{1}{2X + 3Y/4} = 0.622 \quad \text{or} \quad 0.597$$

for cases (i) and (ii), respectively.