ASTR-3760: Solar and Space Physics ....**Problem Set 4** (Due Mon., March 7, 2016)

Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

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1. Stable & unstable stratification of an atmosphere.

(a) Explain qualitatively when the stratification of an atmosphere is stable to convection (use a sketch of specific entropy versus height).

(b) There is a critical temperature gradient, $\beta_{\rm crit} = (dT/dz)_{\rm crit}$, above which the stratification becomes unstable to convection. Show that

$$\beta_{\rm crit} = -\left(1 - \frac{1}{\gamma}\right) g \frac{\mu}{\mathcal{R}},$$

where g is gravity, γ the ratio of specific heats, $\mathcal{R}$ the universal gas constant and μ the mean molecular weight. [Hints: use the condition $c_p^{-1} ds/dz = 0 = \gamma^{-1} d\ln p/dz - d\ln \rho/dz$ for adiabatic stratification, write this in terms of p and T using the perfect gas equation $p = (\mathcal{R}/\mu)\rho T$, and eliminate p using the equation of hydrostatic equilibrium, $dp/dz = -\rho g$.]

(c) Consider an isothermal model of the upper layers of the Sun. Estimate the scale height $H = c_s^2/g$, using $c_s = 7\,{\rm km\,s^{-1}}$, and $g = GM/R^2$ (you may take $G = 7\times10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, $M = 2\times10^{30}\,{\rm kg}$, and $R = 7\times10^{8}\,{\rm m}$).

(d) Calculate the entropy gradient

$$\frac{1}{c_p}\frac{ds}{dz} = \left(1 - \frac{1}{\gamma}\right)\frac{1}{H}$$

with $\gamma = 5/3$.

(e) A hot bubble is in pressure equilibrium, but with a 10% density deficit relative to the surroundings. Calculate how far it will rise before reaching equilibrium.

(f) What are buoyancy oscillations (also known as Brunt-Väisälä oscillations)? What is the relevant restoring force? Give a mathematical expression of the force per unit mass, i.e., the acceleration.

(g) Estimate (within a factor of 3) the period $T_{\rm BV} = 2\pi/\omega_{\rm BV}$, where

$$\omega_{\rm BV} = \left(1 - \frac{1}{\gamma}\right)^{1/2} \frac{g}{c_s}, \qquad (1)$$

for the solar atmosphere (radius from the center $r = 700\,{\rm Mm}$, sound speed $c_{\rm s} = 6\,{\rm km\,s^{-1}}$) and the bottom of the solar convection zone ($r = 500\,{\rm Mm}$, $c_{\rm s} = 200\,{\rm km\,s^{-1}}$). Use $\gamma = 5/3$, $g = GM/R^2$, where $G = 7\times10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, and $M = 2\times10^{30}\,{\rm kg}$. Give the period in seconds, minutes, or hours, as appropriate.

2. Momentum and energy equations in conservative forms. Consider the continuity, momentum, and energy equations in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

and

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p = 0, \tag{3}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} e + \frac{p}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{4}$$

where e is the internal energy per unit mass (which was called u in the lecture).

(a) Derive the evolution equation for the momentum density

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij} p) \tag{5}$$

Note that summation over double indices is assumed!

(b) Explain why this equation is in *conservative* form. Discuss how the volume-integrated momentum changes for periodic boundary conditions. What other boundary conditions give the same result?

(c) Derive the so-called total energy equation in the form

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j \left(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e + p\right)\right], \tag{6}$$

Again, summation over double indices is assumed.

(d) Explain in words how these equations can be used to say something about hydrodnamic planar shocks, where density, pressure, and density can discontinuously across a surface. Consider a one-dimensional frame of reference *comoving* with the shock. What happens to the time derivative in that frame? Use the equation of state in the form

$$P = (\gamma - 1)\rho e$$

and count how many unknowns do you have?

3. Sound waves in a stratified atmosphere. Consider the continuity and momentum equations for an isothermal atmosphere with constant speed of sound, c_s, and uniform gravity, g, in one dimension,

$$\frac{\partial \rho}{\partial t} + u_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial u_z}{\partial z} = 0, \tag{7}$$

$$\rho \frac{\partial u_z}{\partial t} + \rho u_z \frac{\partial u_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \tag{8}$$

where ρ is density and u_z vertical velocity. Instead of using subscripts 0 and 1 for equilibrium and perturbed solutions, as we did in the lecture, we use here overbars and primes instead. Here, the quantities with an overbar are not necessarily constant. By contrast, the subscipts 0 refers now to a constant coefficient of the equilibrium solution and subscipts 1 denotes a constant coefficient in the perturbed solution. Consult lecture 11, page 12, for a similar (but simpler) problem.

(a) Show that the solution for hydrostatic equilibrium, $u_z = \overline{u_z} = 0$, is $\rho = \overline{\rho}(z) = \rho_0 \, e^{-z/H}$, where ρ_0 is a constant and $H = c_s^2/g$ is the vertical scale height.

(b) Write $\rho = \overline{\rho} + \rho'$ and $u_z = u_z'$ and linearise equations (7) and (8) with respect to ρ' and u_z'.

(c) Assume that ρ' and $u_z = u_z'$ take the form

$$\rho'(z,t) = \rho_1 e^{ikz - i\omega t - z/2H}, \tag{9}$$

$$u_z'(z,t) = w_1 e^{ikz - i\omega t + z/2H}, \tag{10}$$

and show that the linearised equations can be written as

$$\begin{pmatrix} -i\omega & [ik - (2H)^{-1}] \\ [ik + (2H)^{-1}]c_s^2 & -i\omega \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_0 w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

(d) Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.

(e) Give a qualitative plot of the dispersion relation.

(f) Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6\,\mathrm{km/s}$ and $g = 270\,\mathrm{ms}^2$.