ASTR-3760: Solar and Space Physics ... Problem Set 5 KEY (Due Mon., April 11, 2016)

Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

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1. Your favorite space weather story.

(a) Describe in a few words your favorite space weather story.

(b) Try to complement the story with some simple numerical estimates to check whether it makes sense.

(c) How frequent is an event of the type you described?

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(a) My favorite space weather story is the event of November 4, 2015, when at 14:00 UT most of Sweden's aviation radars experienced heavy disturbances that subsequently led to all flight radars to be turned off for about two hours. In the beginning, space whether was made responsible for this, although it was unusual that there was only a medium sized M3.7 flare on the Sun, which did not trigger any immediate warnings. However, Opgenoorth et al.[1] reported that this event was actually caused by an *extreme radio-burst* with a flux density of more than 50,000 solar flux units (sfu, and 1 sfu $= 10^{-22}\,\mathrm{W\,m^{-2}\,Hz^{-1}} = 10^4\,\mathrm{Jy}$) at GHz frequencies.

(b) A typical radio frequency for solar observations is 10 cm. In fact, the emission at 10.7 cm ($= 2.8\,\mathrm{GHz}$) is used by NOAA as an indicator of solar activity and is called the F10.7 index. Looking at Fig. 1.6 of Stix (see also Lecture 2, page 20), we see that at that wavelength, the thermal background flux is $10^{-20}\,\mathrm{W\,m^{-2}\,Hz^{-1}}$, which corresponds to 100 sfu. The brightness temperature at that wavelength is slightly above chromospheric values, i.e. $10,000\,\mathrm{K}$. An increase by a factor of 500 would mean 5 million Kelvin, which is comparable to the Virial temperature, above which the thermal velocity would exceed the escape speed ($c_\mathrm{p}T \approx GM/R$, which gives `7e-11*2e30/7e8/(2.5*8300)` $\mathrm{K} = 10^7\,\mathrm{K}$).

(c) To judge how frequent such events are, I looked for earlier studies measuring solar radio flares. Nita et al.[2] studied 412 flares during 2001–2002 at frequencies in the range 1–18 GHz and found that the number density of flares $n(S)$ decreases with increasing flux density S like a powerlaw with $n(S) \approx 2600 S^{-1.7}$. According to Table 2 of their paper, an event with $S = 100\,\mathrm{sfu}$ occurs roughly once or twice a day. An event with $S = 5 \times 10^4\,\mathrm{sfu}$ has a 500 times larger flux density, and since $500^1.7 \approx 4 \times 10^4$, one may conclude that even event like that in Sweden could occur roughly once every 10 years.

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2. Eddington approximation.

In class we used several approximations to derive the temperature structure $T(\tau)$ of the solar

[1] Opgenoorth et al., 2016, "Solar activity during the space weather incident of Nov 4., 2015 – Complex data and lessons learned," Geophys. Res. Abstr., Vol. 18, EGU2016-12017

[2] Nita, G. M., Gary, D. E., & Lee, J., 2004, "Statistical Study of Two Years of Solar Flare Radio Spectra Obtained with the Owens Valley Solar Array" Astrophys. J., 605, 528–545

photosphere. If one uses this to solve for the source function S, and then plug S back into the equation of radiative transfer, we get a differential equation for $I(\mu,\tau)$. The solution to that equation is:

$$I(\mu,\tau) \;=\; \begin{cases} 3H_\odot\,(\tau+\mu+q) & \text{for } \mu>0 \text{ (i.e., upward propagating rays),} \\ 3H_\odot\left[(\tau+\mu+q)-(\mu+q)\,e^{\tau/\mu}\right] & \text{for } \mu<0 \text{ (i.e., downward propagating rays).} \end{cases}$$

where $q=2/3$ and $H_\odot$ is proportional to the constant energy flux of radiation through the atmosphere. Also, let's define

$$I_{\rm up}(\tau) \;=\; I(+1,\tau) \qquad \text{and} \qquad I_{\rm down}(\tau) \;=\; I(-1,\tau)$$

(a) For the limiting case of $\tau \gg 1$, write simpler approximations for $I_{\rm up}$ and $I_{\rm down}$. Compute the ratio $I_{\rm up}/I_{\rm down}$. Does it make sense for this limit of the "deep interior?"

(b) Write the angle dependence $I(\mu)$ for the limiting case of $\tau \to 0$. Does it agree with what was discussed in class about the phenomenon of "limb darkening" at the solar surface?

(c) At the surface ($\tau=0$), compute the angle moments

$$J=\frac{1}{2}\int_{-1}^{+1} d\mu\; I(\mu) \qquad\qquad H=\frac{1}{2}\int_{-1}^{+1} d\mu\;\mu\; I(\mu) \qquad\qquad K=\frac{1}{2}\int_{-1}^{+1} d\mu\;\mu^2\; I(\mu)$$

as functions of $H_\odot$. *Note:* You can set aside (for now) the fact that the moment quantity H is supposed to be the same thing as $H_\odot$. We're trying to see how "self-consistent" this model is.

(d) From your answer to part (c), evaluate whether Eddington's approximations ($J=3K$ and $J=2H$) are good approximations at the solar surface. In other words, if these approximations are not exactly true, then compute how "bad" they are (i.e., percentage error). You can also compare H to $H_\odot$ to get another estimate of the error.

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(a) For arbitrary values of τ, the up/down intensities can be written

$$I_{\rm up} = 3H_\odot(\tau+1+\tfrac{2}{3}) = 3H_\odot(\tau+\tfrac{5}{3}) = H_\odot(3\tau+5)$$

$$I_{\rm down} = 3H_\odot[(\tau-1+\tfrac{2}{3})-(-1+\tfrac{2}{3})\,e^{-\tau} = 3H_\odot(\tau-\tfrac{1}{3}+\tfrac{1}{3}\,e^{-\tau}) \approx H_\odot(3\tau-1)$$

(Remember that for the "downward" case that μ is negative!) Thus, in the limiting case of the deep atmosphere ($\tau \gg 1$), the dominant terms are

$$I_{\rm up} \;\approx\; 3H_\odot\tau \qquad \text{and} \qquad I_{\rm down} \;\approx\; 3H_\odot\tau$$

so that the ratio $I_{\rm up}/I_{\rm down} \approx 1$. This makes sense, since as you go deeper into the interior, the radiation field becomes more and more an **isotropic** blackbody. Anisotropy remains (because the flux is finite), but it becomes weaker in relative terms. More accurately, we can write

$$I_{\rm up}/I_{\rm down} \approx (\tau+\tfrac{5}{3})/(\tau-\tfrac{1}{3}) \approx (1+\tfrac{5}{3}\tau^{-1})/(1-\tfrac{1}{3}\tau^{-1}) \approx (1+\tfrac{5}{3}\tau^{-1})(1+\tfrac{1}{3}\tau^{-1}) \approx 1+2\tau^{-1}.$$

Figure 1 confirms that $I_{\rm up}/I_{\rm down}$ approaches unity like $1+2\tau^{-1}$.

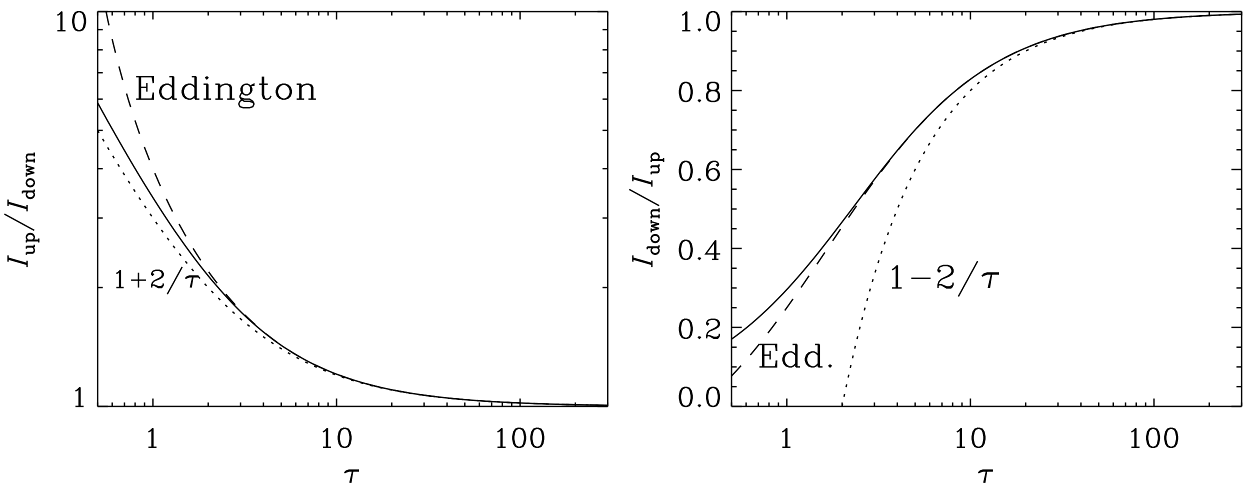

Figure 1: Double-logarithmic representation of the τ dependence of the ratio $I_{\rm up}/I_{\rm down}$ and semi-logarithmic representation of the inverse ratio $I_{\rm down}/I_{\rm up}$. For $\tau > 10$, the ratio is close to $1 + 2\tau^{-1}$, while for $\tau < 2$, the Eddington approximation (shown in dashed) begins to deviate.

(b) In the limit of $\tau = 0$, the general expression for the intensity is

$$I(\mu, 0) = \begin{cases} 3H_\odot \left(\mu + \frac{2}{3}\right) & \text{for } \mu > 0, \\ 0 & \text{for } \mu < 0. \end{cases}$$

This makes sense. At the top of the atmosphere, there are no rays pointing inward. The rays coming up from below exhibit limb darkening, since the center of the Sun ($\mu = +1$) has an intensity of $5H_\odot$ while the grazing rays from the limb of the Sun ($\mu = 0$) have an intensity of $2H_\odot$. The limb is about 40% less bright than the disk-center.

(c) We want to integrate the equation given in the answer to part (b) above, which means the integration limits only need to go from 0 to +1. After going through the integrations, we get

$$J = \tfrac{1}{2}\int_0^1 I(\mu,0)\,{\rm d}\mu = \frac{3H_\odot}{2}\int_0^1 (\mu + \tfrac{2}{3})\,{\rm d}\mu = H_\odot(\tfrac{3}{4} + 1) = \frac{7}{4}H_\odot = \frac{7}{4}\frac{F_\odot}{4\pi} \tag{1}$$

$$H = \tfrac{1}{2}\int_0^1 I(\mu,0)\mu\,{\rm d}\mu = \frac{3H_\odot}{2}\int_0^1 (\mu^2 + \tfrac{2}{3}\mu)\,{\rm d}\mu = 3H_\odot(\tfrac{1}{3} + \tfrac{1}{3}) = H_\odot = \frac{F_\odot}{4\pi} \tag{2}$$

$$K = \tfrac{1}{2}\int_0^1 I(\mu,0)\mu^2\,{\rm d}\mu = \frac{3H_\odot}{2}\int_0^1 (\mu^3 + \tfrac{2}{3}\mu^2)\,{\rm d}\mu = 3H_\odot(\tfrac{1}{4} + \tfrac{2}{9}) = H_\odot(\tfrac{3}{8} + \tfrac{1}{3}) = H_\odot\tfrac{9+8}{24} = H_\odot\tfrac{17}{24} = \tfrac{17}{24}\frac{F_\odot}{4\pi} \tag{3}$$

(d) It turns out that we didn't have to worry about the possibility that H was different from $H_\odot$. However, we do have to worry about departures from Eddington's two approximations. Instead of $J = 2H$, we have $J = 1.75\,H$. If we treat the coefficient 1.75 as the correct value and the coefficient of 2 as the "guess," the percentage error of that guess is

$$\frac{2 - 1.75}{1.75} \approx 14\% \; .$$

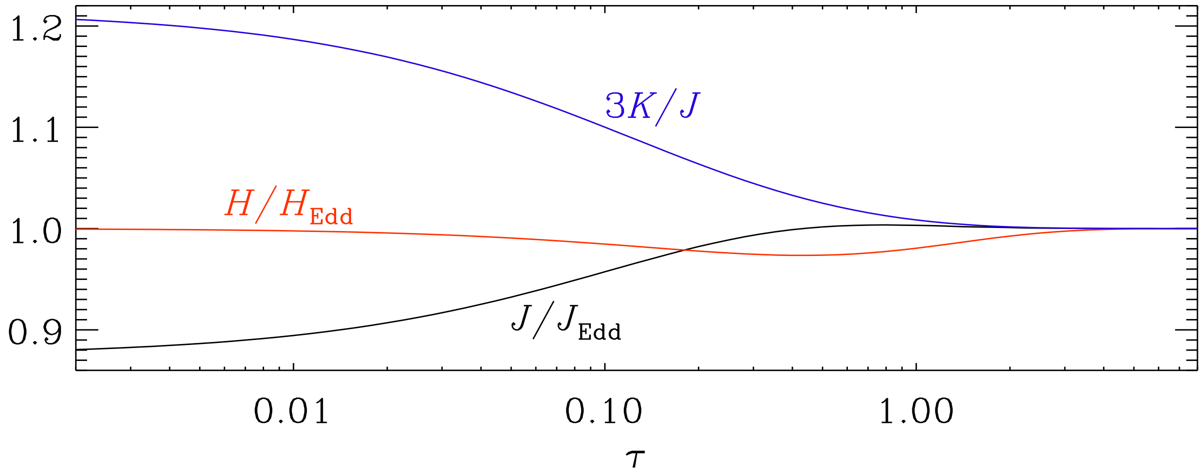

Figure 2: Eddington approximation at different optical depth τ.

Instead of $J = 3K$, we have

$$J = \frac{7}{4}H = \frac{7}{4}\left(\frac{24}{17}K\right) = \frac{42}{17}K \approx 2.471\,K \ .$$

Thus, if we treat the coefficient 2.471 as the actual value and the coefficient of 3 as the guess, the percentage error of that guess is

$$\frac{3 - 2.471}{2.471} \approx 21\% \ .$$

Thus, the Eddington approximations are okay if we are willing to ignore inconsistencies at the 15–20% level.

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2. Is the solar corona in hydrostatic equilibrium? In class, we will derive that a hot corona (with $T \sim 10^6$ K) will be dominated by heat conduction in the regions far above the surface. In these regions, the temperature drops off very slowly with increasing distance,

$$T(r) = T_0 \left(\frac{r_0}{r}\right)^{2/7}$$

where r_0 is a base radius in the corona, often assumed to be about 2 $R_\odot$, and T_0 is the temperature at the base radius.

(a) Assuming a corona with no fluid flow (i.e., $\mathbf{u} = 0$ everywhere) in spherical symmetry, write the equation of hydrostatic equilibrium and show that it can be simplified into the form

$$\frac{d}{dr}\left(\frac{\rho}{r^{2/7}}\right) = -C_1 \frac{\rho}{r^2}$$

and give an expression for the constant C_1 in terms of the solar mass, the properties at r_0, and other physical constants.

(b) Show that the following solution satisfies the above differential equation,

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{2/7} \exp\left\{ C_2 \left[\left(\frac{r_0}{r}\right)^{5/7} - 1 \right] \right\} \qquad \text{where} \quad C_2 = \frac{7\,C_1}{5\,r_0^{5/7}}$$

and ρ_0 is the density at r_0.

(c) If $r_0 = 2\,R_\odot$, $T_0 = 2 \times 10^6$ K, and $\rho_0 = 2 \times 10^{-15}$ kg m^{-3}, then solve the above expression for the *hydrogen number density* $n_{\rm H}$ at a distance of 1 AU. Feel free to assume that the corona is all hydrogen (i.e., just protons and electrons).

(d) The observed number density at 1 AU is usually between 1 and 10 protons per cm^3. Is the hydrostatic model a good one?

(e) Write an expression for the gas pressure P using the above hydrostatic model. Show that as $r \to \infty$, P approaches a constant value (call it P_∞) and derive an expression for P_∞.

(f) Compute a value for P_∞ given the constants from part (c).

(g) Astronomers find that the gas pressure in the interstellar medium (far outside the influence of the Sun) is about 10^{-14} to 10^{-13} pascals (i.e., N m^{-2}). Again, do you think this hydrostatic model is realistic? What do you think is *really* happening?

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(a) In spherical symmetry, the equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho\, g \qquad \text{where, above the Sun's surface,} \qquad g = \frac{GM_\odot}{r^2} .$$

Using the ideal gas law, we can express P as a product of ρ and T, with

$$\frac{dP}{dr} = \frac{k_{\rm B}}{\mu m_{\rm H}} \frac{d}{dr}(\rho\, T) = \frac{k_{\rm B} T_0\, r_0^{2/7}}{\mu m_{\rm H}} \frac{d}{dr}\left(\frac{\rho}{r^{2/7}}\right)$$

Thus, we can collect the constants on the right-hand side,

$$\frac{d}{dr}\left(\frac{\rho}{r^{2/7}}\right) = -\left(\frac{GM_\odot\, \mu m_{\rm H}}{k_{\rm B} T_0\, r_0^{2/7}}\right) \frac{\rho}{r^2} \qquad \text{i.e.,} \qquad C_1 = \frac{GM_\odot\, \mu m_{\rm H}}{k_{\rm B} T_0\, r_0^{2/7}}$$

(b) We have to show that the left-hand side of the differential equation equals the right-hand side.

$$\text{LHS} = \frac{d}{dr}\left(\rho\, r^{-2/7}\right) = r^{-2/7} \frac{d\rho}{dr} - \frac{2}{7} r^{-9/7} \rho \qquad \text{and} \qquad \text{RHS} = -\frac{C_1\, \rho}{r^2} .$$

To help keep everything easy to manage, let's write

$$\rho = \frac{\rho_0\, r^{2/7}\, e^{f(r)}}{r_0^{2/7}} \qquad \text{where} \qquad f(r) = C_2 \left(\frac{r_0}{r}\right)^{5/7} - C_2 .$$

To evaluate LHS, we need to take the derivative $d\rho/dr$, which ends up depending on the derivative df/dr.

$$\frac{d\rho}{dr} = \frac{\rho_0 e^{f(r)}}{r_0^{2/7}} \left(\frac{2}{7\, r^{5/7}} + r^{2/7} \frac{df}{dr} \right) \qquad \text{with} \qquad \frac{df}{dr} = -\frac{5\, C_2\, r_0^{5/7}}{7\, r^{12/7}} .$$

From the above, we get $$\frac{d\rho}{dr} = \rho_0 e^{f(r)} \left(\frac{2}{7\,r_0^{2/7}\,r^{5/7}} - \frac{5\,C_2\,r_0^{3/7}}{7\,r^{10/7}} \right) .$$

Putting this back into LHS gives three terms, two of which cancel out with one another:

$$\text{LHS} = \rho_0 e^{f(r)} \left(\frac{2}{7\,r_0^{2/7}\,r} - \frac{5\,C_2\,r_0^{3/7}}{7\,r^{12/7}} - \frac{2}{7\,r_0^{2/7}\,r} \right)$$

The remaining term is very similar to what we get when we write out RHS,

$$\text{RHS} = \rho_0 e^{f(r)} \left(\frac{-C_1}{r_0^{2/7} r^{12/7}} \right)$$

and using the above definition for C_2 (in terms of C_1 and r_0) shows us that LHS = RHS.

(c) First, let's evaluate the constant C_2, which is dimensionless:

$$C_2 = \frac{7\,G M_\odot\,\mu m_{\rm H}}{5\,k_{\rm B} T_0\,r_0} = 4.04 .$$

The radial distance of 1 AU corresponds to $r = 215\,R_\odot$, so the ratio $(r/r_0) = 107.5$ is also a dimensionless number. Plugging them in, we get

$$\rho = 0.077\,\rho_0 \approx 1.5 \times 10^{-16} \text{ kg m}^{-3} .$$

For a pure hydrogen plasma, $\rho = m_{\rm H} n_{\rm H}$, so we can solve for $n_{\rm H} \approx 9 \times 10^{10}$ protons / m^3.

(d) Space physicists like to report quantities in cgs units. The range of measured number densities $n = 1$–10 cm^{-3} is equivalent to $n = 10^6$–10^7 m^{-3}. This is 4 or 5 orders of magnitude *lower* than the value we computed in part (c). Thus, it's looking like the hydrostatic model has some serious problems.

(e) Plugging in the solutions for ρ and T, we get

$$P = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}} = \frac{k_{\rm B}}{\mu m_{\rm H}} \left[\rho_0 \left(\frac{r}{r_0} \right)^{2/7} e^{f(r)} \right] \left[T_0 \left(\frac{r_0}{r} \right)^{2/7} \right] = \frac{\rho_0 k_{\rm B} T_0}{\mu m_{\rm H}}\, e^{f(r)} .$$

The $(r/r_0)^{2/7}$ factors cancel out with one another. The only remaining r dependence in $P(r)$ is inside the function $f(r)$. However, as we take the limit $r \to \infty$, we see that $f(r)$ approaches a constant value of $-C_2$. Thus,

$$P_\infty = \frac{\rho_0 k_{\rm B} T_0}{\mu m_{\rm H}} \exp\left(-\frac{7\,G M_\odot\,\mu m_{\rm H}}{5\,k_{\rm B} T_0\,r_0} \right)$$

(f) Plugging in the numbers, we get $\boxed{P_\infty \approx 10^{-6} \text{ pascals}}$.

(g) Just like in part (d), it looks like the hydrostatic model greatly over-estimates the plasma properties in the distant heliosphere. Our computed P_∞ is 7 to 8 orders of magnitude larger than the interstellar gas pressure. If the corona was really hydrostatic, there would be a *huge* gas pressure gradient that would cause the corona to expand rapidly into the ISM, overpowering it out to distances of several parsecs! This unrealistic prediction is one piece of evidence that led scientists to think about the possibility of an outflowing **solar wind** (whose pressure $P(r)$ keeps decreasing as r increases) instead of a static plasma.