Eddington approximation

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In lecture 26, page 4, we derived the solution to the Eddington approximation as

$$I(\tau,\mu) = \tfrac{3}{4\pi} F\,(\tau + \tfrac{2}{3} + \mu), \tag{1}$$

where F is the flux. The first three moments are

$$J(\tau) = \tfrac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mathrm{d}\mu = \tfrac{3}{4\pi}F\,\tfrac{1}{2}\int_{-1}^{1}\left[(\tau+\tfrac{2}{3})+\mu\right]\mathrm{d}\mu = \tfrac{3}{4\pi}F\,(\tau+\tfrac{2}{3}), \tag{2}$$

$$H(\tau) = \tfrac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mu\,\mathrm{d}\mu = \tfrac{3}{4\pi}F\,\tfrac{1}{2}\int_{-1}^{1}\left[(\tau+\tfrac{2}{3})\mu+\mu^2\right]\mathrm{d}\mu = \tfrac{1}{4\pi}F \tag{3}$$

$$K(\tau) = \tfrac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mu^2\,\mathrm{d}\mu = \tfrac{3}{4\pi}F\,\tfrac{1}{2}\int_{-1}^{1}\left[(\tau+\tfrac{2}{3})\mu^2+\mu^3\right]\mathrm{d}\mu = \tfrac{1}{4\pi}F\,(\tau+\tfrac{2}{3}), \tag{4}$$

Thus, we see that $3K(\tau) = J(\tau) = (3/4\pi)F\,(\tau+\frac{2}{3})$ for all τ and $J(0) = 2H(\tau)$. In thermal equilibrium, the source function is always equal to the mean intensity, i.e., $S(\tau) = J(\tau)$.

Comparison with the formal solution

In Homework 2(b), we verified that

$$I(\tau,\mu) = I(\tau_0,\mu)\,e^{-(\tau_0-\tau)/\mu} + \int_{\tau}^{\tau_0} S(\tau')\,e^{-(\tau'-\tau)/\mu}\,d\tau'/\mu \tag{5}$$

which is known as the *formal solution* of the radiative transfer equation (see Stix, p. 150). We are now interested in two special cases: (i) upward propagating rays ($\mu > 0$) that receive radiation from all the way down to $\tau_0 \to \infty$, and (ii) downward propagating rays ($\mu < 0$) that receive radiation from all the way up to $\tau_0 = 0$ with $I(\tau_0,\mu) = 0$, i.e., there is no illumination from the top. Thus, we have

$$I(\tau,\mu) = e^{\tau/\mu}\int_{\tau}^{\infty} S(\tau')\,e^{-\tau'/\mu}\,d\tau'/\mu \quad \text{(for upward rays, } \mu > 0), \tag{6}$$

$$I(\tau,\mu) = e^{\tau/\mu}\int_{\tau}^{0} S(\tau')\,e^{-\tau'/\mu}\,d\tau'/\mu \quad \text{(for downward rays, } \mu < 0). \tag{7}$$

Inserting now the Eddington solution, $S(\tau) = J(\tau) = (3/4\pi)F\,(\tau+\frac{2}{3})$, we find[1]

$$I(\tau,\mu) = (3/4\pi)\,F\,(\tau+\tfrac{2}{3}+\mu) \qquad \text{(for } \mu > 0), \tag{8}$$

$$I(\tau,\mu) = (3/4\pi)F\left[(\tau+\tfrac{2}{3}+\mu) - (\tfrac{2}{3}+\mu)\,e^{\tau/\mu}\right] \qquad \text{(for } \mu < 0). \tag{9}$$

so it agrees with the solution to the Eddington approximation exept for an additional term, $-(\frac{2}{3}+\mu)\,e^{\tau/\mu}$, for the downward propagating rays; see Fig. 1. Without this "correction term", $I_{\rm dn} \equiv I(\tau,-1)$ would actually become negative for $\tau < 1/3$, which is unphysical. With the correction term included, we have for $\mu = -1$

$$I_{\rm dn}/[(3/4\pi)F] = \tau - \tfrac{1}{3} + \tfrac{1}{3}e^{-\tau} = \tau - \tfrac{1}{3}(1-e^{-\tau}) \approx \tau - \tfrac{1}{3}\tau = 2\tau/3 \quad \text{(for } \tau \to 0). \tag{10}$$

[1] Details regarding the derivation are not relevant now, but the calculation is similar to that in the key to Homework 2, problem 1.

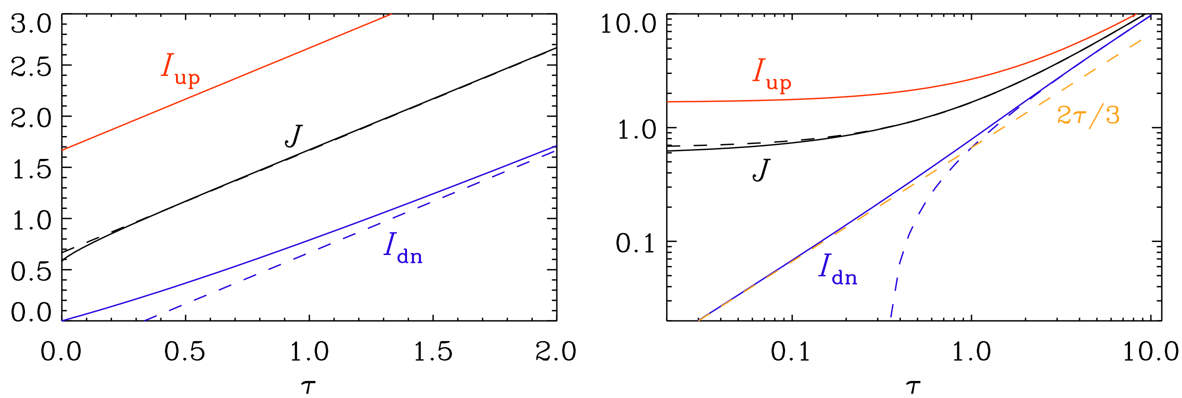

Figure 1: Linear and logarithmic representations of the τ dependencies of the intensities of upward ($I_{\rm up}$, red) and downward ($I_{\rm dn}$, blue) propagating rays, compared with the mean intensity J (black). The solution to the Eddington approximation is shown in dashed. Here, $(3/4\pi)F = 1$ is assumed.

Remarks regarding Homework 5, problem 1

(a) the expression for $I_{\rm dn} \equiv I(\tau, -1)$ is given in Eq. (10) as $I_{\rm dn} \propto \tau - \frac{1}{3}(1 - e^{-\tau})$. The expression for $I_{\rm up} \equiv I(\tau, +1)$ is even simpler.

(b) the relevant expression for $\tau = 0$ is just $I(\mu) = \frac{3}{4\pi} F \left(\frac{2}{3} + \mu\right) = 3H \left(\frac{2}{3} + \mu\right)$.

(c) the result of a numerical integration is shown in Fig. 2 as a function of τ. For $\tau = 0$, the integration is quite straightforward.

(d) the results can be read off Fig. 2, but here you are supposed to compute actual values. [Under (d), it should of course read "From your answer to part (c), ...]

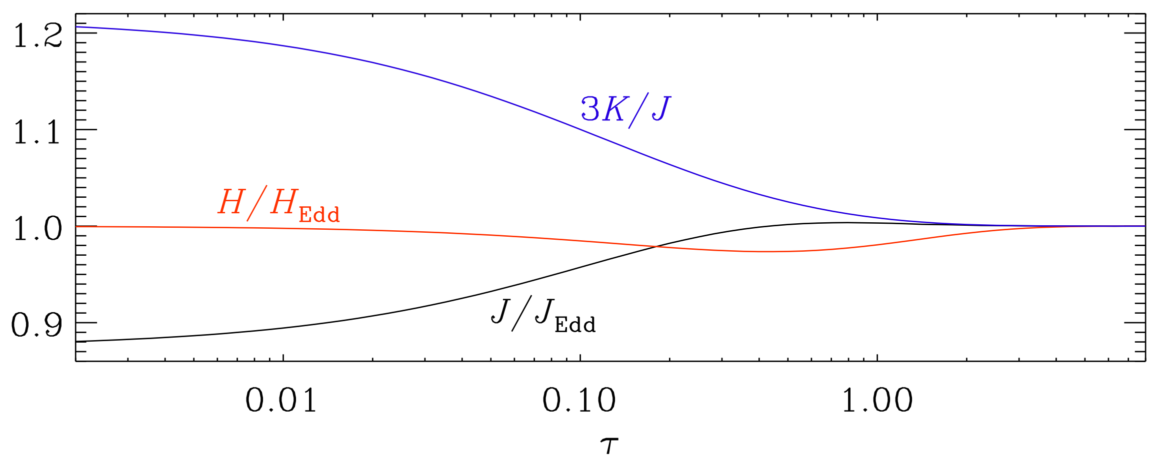

Figure 2: Eddington approximation at different optical depth τ.

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