Problems

1. There is a critical temperature gradient, $\beta_{\rm crit} = (dT/dz)_{\rm crit}$, above which the stratification becomes unstable to convection. Show that

$$\beta_{\rm crit} = -\left(1 - \frac{1}{\gamma}\right) g \frac{\mu}{\mathcal{R}},$$

where g is gravity, γ the ratio of specific heats, $\mathcal{R}$ the universal gas constant and μ the mean molecular weight. [Hints: use the condition $c_p^{-1} ds/dz = 0 = \gamma^{-1} d\ln p/dz - d\ln\rho/dz$ for adiabatic stratification, write this in terms of p and T using the perfect gas equation $p = (\mathcal{R}/\mu)\rho T$, and eliminate p using the equation of hydrostatic equilibrium, $dp/dz = -\rho g$.]

[10 marks]

2. What are Brunt-Väisälä oscillations? What are the relevant restoring forces? Estimate, to an order of magnitude, the period $T_{\rm BV} = 2\pi/\omega_{\rm BV}$ in years, where

$$\omega_{\rm BV} = \left(1 - \frac{1}{\gamma}\right)^{1/2} \frac{g}{c_s}, \tag{1}$$

for a cluster of galaxies at a radius R from the centre. Take $\gamma = 5/3$, $g = GM/R^2$, where $G = 7 \times 10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, $M = 10^{44}\,{\rm kg}$, and $R = 300\,{\rm kpc}$. Use $c_s^2 = GM/R$. Note that $1\,{\rm kpc} = 3 \times 10^{19}\,{\rm m}$.

[10 marks]

3. Use the continuity, Euler, and energy equations

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{v}) = 0, \tag{2}$$

and

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla} p = 0, \tag{3}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} e + \frac{p}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{4}$$

to derive the following equations

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_j}(\rho v_i v_j + \delta_{ij} p) \tag{5}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{v}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[v_j\left(\tfrac{1}{2}\rho\boldsymbol{v}^2 + \rho e + p\right)\right] \tag{6}$$

Note that summation over double indices is assumed!

[10 marks]

4. Consider the continuity and momentum equations for an isothermal atmosphere with constant speed of sound, c_s, and uniform gravity, g, in one dimension,

$$\frac{\partial \rho}{\partial t} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_z}{\partial z} = 0, \tag{7}$$

$$\rho \frac{\partial v_z}{\partial t} + \rho v_z \frac{\partial v_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \tag{8}$$

where ρ is density and v_z vertical velocity.

(a) Show that the solution for hydrostatic equilibrium, $v_z = \overline{v_z} = 0$, is $\rho = \overline{\rho}(z) = \rho_0\, e^{-z/H}$, where ρ_0 is a constant and $H = c_s^2/g$ is the vertical scale height.

(b) Write $\rho = \overline{\rho} + \rho'$ and $v_z = v_z'$ and linearise equations (**??**) and (**??**) with respect to ρ' and v_z'.

(c) Assume that ρ' and $v_z = v_z'$ take the form

$$\rho'(z,t) = \rho_1 e^{ikz - i\omega t - z/2H}, \tag{9}$$

$$v_z'(z,t) = w_1 e^{ikz - i\omega t + z/2H}, \tag{10}$$

and show that the linearised equations can be written as

$$\begin{pmatrix} -i\omega & [ik - (2H)^{-1}] \\ [ik + (2H)^{-1}]c_s^2 & -i\omega \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_0 w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

(d) Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.

(e) Give a qualitative plot of the dispersion relation.

(f) Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6\,\mathrm{km/s}$ and $g = 270\,\mathrm{ms}^2$.

[30 marks]