Problems

1. For a steady spherically symmetric flow of an isothermal gas with constant sound speed $c_{\rm s}$, the Euler equation with a suitable body force is

$$u_r \frac{{\rm d}u_r}{{\rm d}r} = -c_{\rm s}^2 \frac{{\rm d}\ln\rho}{{\rm d}r} - \frac{GM}{r^2}. \tag{1}$$

The continuity equation is

$$\frac{1}{r^2}\frac{\rm d}{{\rm d}r}(r^2 \rho u_r) = 0. \tag{2}$$

(a) Show that the critical radius r_* where $|u_r| = c_{\rm s}$, is $r_* = GM/(2c_{\rm s}^2)$.

(b) Integrate Eqs (1) and (2), and then eliminate $\ln\rho$ to show that

$$\tfrac{1}{2}u_r^2 - c_{\rm s}^2 \ln|u_r| - 2c_{\rm s}^2 \ln r - \frac{GM}{r} = \tfrac{1}{2}c_{\rm s}^2 - c_{\rm s}^2 \ln c_{\rm s} - 2c_{\rm s}^2 \ln r_* - \frac{GM}{r_*}. \tag{3}$$

(c) Show that Eq. (3) can be written as

$$\mathcal{M} = \sqrt{C + 2\ln\mathcal{M}}, \tag{4}$$

where $\mathcal{M} = |u_r|/c_{\rm s}$ and

$$C = 4\left(\ln\tilde{r} + \frac{1}{\tilde{r}}\right) - 3,$$

with $\tilde{r} = r/r_*$.

(d) Calculate the value of C for $\tilde{r} = 10$, and find the corresponding value of $\mathcal{M}$ using three iteration steps starting with $\mathcal{M} = 1$. Show your working in all intermediate steps. Sketch the solution for $\mathcal{M}$ against $\tilde{r}$, and indicate the points where $\tilde{r} = 1$ and 10.

(e) Show that Eq. (4) can also be written as $\mathcal{M} = \exp\left[\frac{1}{2}(\mathcal{M}^2 - C)\right]$, and, for the same value of C, iterate for $\mathcal{M}$ starting again with $\mathcal{M} = 1$ (use three iterations, show your working). Again, sketch the solution of $\mathcal{M}$ against $\tilde{r}$, indicate the points where $\tilde{r} = 1$ and 10, and show the direction of the flow. In what area of stellar physics can this model be applied?

[30 marks]

2. Use dimensional arguments to determine the form of the energy spectrum $E(k)$ for hydromagnetic turbulence. You may assume that the spectrum can be written in the form

$$E(k) = C\,(v_A\epsilon)^a k^b,$$

where C is a dimensionless constant, v_A is the Alfvén speed, ϵ (with dimension $\mathrm{m^2\,s^{-3}}$) the energy injection rate, and k the wavenumber.

[Note that $\int E(k)dk$ has the dimension $\mathrm{m^2\,s^{-2}}$.]

[10 marks]

3. (a) Start from $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})}$ and show that

$$\dot{\mathcal{E}}_i^{\mathrm{K}} = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\overline{u_j\partial_l u_n\overline{B}_p} \tag{1}$$

$$\dot{\mathcal{E}}_i^{\mathrm{K}} = \alpha_{ip}^{\mathrm{K}}\overline{B}_p + \eta_{ipl}^{\mathrm{K}}\overline{B}_{p,l} \tag{2}$$

(b) Show that

$$\alpha_{ip}^{\mathrm{K}} = \epsilon_{jnp}\overline{u_j u_{n,i}} - \epsilon_{inp}\overline{u_j u_{n,j}} \tag{3}$$

and

$$\eta_{ipl}^{\mathrm{K}} = -\epsilon_{inp}\overline{u_l u_n} \tag{4}$$

(c) Assume isotropy: $\alpha_{\mathrm{K}} = \frac{1}{3}\delta_{ip}\alpha_{ip}^{\mathrm{K}}$ and show that $\eta_{\mathrm{K}} = \frac{1}{6}\epsilon_{ipl}\eta_{ipl}^{\mathrm{K}}$, so

$$\alpha_{\mathrm{K}} = \tfrac{1}{3}\epsilon_{jni}\overline{u_j u_{n,i}} - \tfrac{1}{3}\epsilon_{ini}\overline{u_j u_{n,j}} = -\tfrac{1}{3}\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}. \tag{5}$$

and

$$\eta_{\mathrm{K}} = -\tfrac{1}{6}(\overline{u_p u_p} - 3\overline{u_n u_n}) = \tfrac{1}{3}\overline{\boldsymbol{u}^2} \tag{6}$$

[30 marks]