Astrophysical Fluid Dynamics: Solutions to Problem Sheet 1

There is a critical temperature gradient, $\beta_{\rm crit} = (dT/dz)_{\rm crit}$, above which the stratification becomes unstable to convection. Show that

$$\beta_{\rm crit} = -\left(1 - \frac{1}{\gamma}\right) g \frac{\mu}{\mathcal{R}},$$

where g is gravity, γ the ratio of specific heats, $\mathcal{R}$ the universal gas constant and μ the mean molecular weight. [Hints: use the condition $c_p^{-1} ds/dz = 0 = \gamma^{-1} d\ln p/dz - d\ln\rho/dz$ for adiabatic stratification, write this in terms of p and T using the perfect gas equation $p = (\mathcal{R}/\mu)\rho T$, and eliminate p using the equation of hydrostatic equilibrium, $dp/dz = -\rho g$.]

• Since the entropy s is constant we have

$$0 = \frac{1}{\gamma}\frac{d\ln p}{dz} - \frac{d\ln\varrho}{dz} = \frac{1}{\gamma}\frac{d\ln p}{dz} - \left(\frac{d\ln p}{dz} - \frac{d\ln T}{dz}\right) = \frac{d\ln T}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{d\ln p}{dz}\ ,$$

because $\varrho \propto p/T$. So we have

$$\frac{d\ln T}{dz} = \left(1 - \frac{1}{\gamma}\right)\frac{d\ln p}{dz}\ . \qquad (1)$$

Using hydrostatic equilibrium, we have

$$\frac{d\ln p}{dz} = \frac{1}{p}\frac{dp}{dz} = -\frac{\varrho}{p} g = -\frac{g}{\mathcal{R}T/\mu}\ . \qquad (2)$$

Combining Eqs. (1) and (2), we have

$$\frac{dT}{dz} = -\left(1 - \frac{1}{\gamma}\right) g \frac{\mu}{\mathcal{R}}\ .$$

Since we did this calculation assuming $s =$ const (adiabaticity) this is then indeed the *critical* temperature gradient, $\beta_{\rm crit}$.

What are Brunt-Väisälä oscillations? What is the relevant restoring force? Estimate, to an order of magnitude, the period $T_{\rm BV} = 2\pi/\omega_{\rm BV}$ in years, where

$$\omega_{\rm BV} = \left(1 - \frac{1}{\gamma}\right)^{1/2} \frac{g}{c_s}, \qquad (3)$$

for a cluster of galaxies at a radius R from the centre. Take $\gamma = 5/3$, $g = GM/R^2$, where $G = 7\times 10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, $M = 10^{44}\,{\rm kg}$, and $R = 300\,{\rm kpc}$. Use $c_s^2 = GM/R$. Note that $1\,{\rm kpc} = 3\times 10^{19}\,{\rm m}$.

• Brunt-Väisälä oscillations are vertical up & down motions in a stably stratified atmosphere. The relevant restoring force is gravity or the buoyancy force, $\delta\rho\, g$.

The Brunt-Väisälä frequency is proportional to

$$\frac{g}{c_{\rm s}} = \frac{GM}{R^2} \times \left(\frac{R}{GM}\right)^{1/2} = \left(\frac{GM}{R^3}\right)^{1/2}.$$

For a galaxy cluster we find

$$\left(\frac{GM}{R^3}\right)^{1/2} = \frac{7\times 10^{-11}\times 10^{44}\,{\rm kg}}{(300\times 3\times 10^{19})^3})\,{\rm s^{-1}} \approx 10^{-16}\,{\rm s^{-1}} \approx 3\times 10^{-9}\,{\rm yr^{-1}}$$

With $1 - 1/\gamma = 0.4$, and hence $\sqrt{1 - 1/\gamma} \approx 0.6$, we have $\omega_{\rm BV} \approx 2\times 10^{-9}\,{\rm yr^{-1}}$. The period is thus 3×10^9 yr.

Use the continuity, Euler, and energy equations

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0, \tag{4}$$

and

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}p = 0, \tag{5}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{6}$$

to derive the following equations

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_j}(\rho v_i v_j + \delta_{ij}p) \tag{7}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{v}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[v_j\left(\tfrac{1}{2}\rho\boldsymbol{v}^2 + \rho e + p\right)\right] \tag{8}$$

Note that summation over double indices is assumed!

• Using the product rule, we have

$$\frac{\partial\rho u_i}{\partial t} = \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial\rho}{\partial t} \tag{9}$$

Likewise

$$\frac{\partial}{\partial x_j}[(\rho u_j)u_i] = (\rho u_j)\frac{\partial}{\partial x_j}u_i + u_i\frac{\partial}{\partial x_j}(\rho u_j) \tag{10}$$

With this we can write

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = \rho\frac{\mathrm{D}u_i}{\mathrm{D}t} + u_i\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) \tag{11}$$

where the last part vanishes owing to the continuity equation, so we just have

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) \tag{12}$$

Again, using the continuity equation, we can write the total derivative of e as

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_j}(\rho u_j e) \tag{13}$$

Next, we compute $\rho\boldsymbol{u}\cdot\mathrm{D}\boldsymbol{u}/\mathrm{D}t$, i.e.,

$$\rho u_i\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{u}^2) + \frac{\partial}{\partial x_j}(\tfrac{1}{2}\rho u_j\boldsymbol{u}^2) \tag{14}$$

This is just mathematics. Next we use the energy and momentum equations, here with some arbitrary flux $\boldsymbol{F}$ added in the energy equation, i.e.,

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = -p\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{15}$$

as well as the momentum equation,

$$\rho u_i\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p \tag{16}$$

Add the two gives

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho\boldsymbol{u}^2) + \frac{\partial}{\partial x_j}(\rho u_j e + \tfrac{1}{2}\rho u_j\boldsymbol{u}^2) = -\boldsymbol{\nabla}\cdot(p\boldsymbol{u} + \boldsymbol{F}) \tag{17}$$

Consider the continuity and momentum equations for an isothermal atmosphere with constant speed of sound, c_s, and uniform gravity, g, in one dimension,

$$\frac{\partial \rho}{\partial t} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_z}{\partial z} = 0, \tag{18}$$

$$\rho \frac{\partial v_z}{\partial t} + \rho v_z \frac{\partial v_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \tag{19}$$

where ρ is density and v_z vertical velocity.

1. Show that the solution for hydrostatic equilibrium, $v_z = \overline{v_z} = 0$, is $\rho = \overline{\rho}(z) = \rho_0 \, e^{-z/H}$, where ρ_0 is a constant and $H = c_s^2/g$ is the vertical scale height.
2. Write $\rho = \overline{\rho} + \rho'$ and $v_z = v_z'$ and linearise equations (18) and (19) with respect to ρ' and v_z'.
3. Assume that ρ' and $v_z = v_z'$ take the form

$$\rho'(z,t) = \rho_1 e^{ikz - i\omega t - z/2H}, \tag{20}$$

$$v_z'(z,t) = w_1 e^{ikz - i\omega t + z/2H}, \tag{21}$$

   and show that the linearised equations can be written as

$$\begin{pmatrix} -i\omega & [ik - (2H)^{-1}] \\ [ik + (2H)^{-1}]c_s^2 & -i\omega \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_0 w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

4. Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.
5. Give a qualitative plot of the dispersion relation.
6. Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6\,\text{km/s}$ and $g = 270\,\text{ms}^2$.

• 1. In hydrostatic equipibrium we have

$$c_s^2 \frac{\partial \ln \rho}{\partial z} = g$$

so $\ln \rho/\rho_0 = -gz/c_s^2$ and therefore $\rho = \rho_0 \exp(-gz/c_s^2)$.
2–4. Need to write down...
5. A plot of the dispersion relation is shown in figure 1.
6. Around 5 minutes.

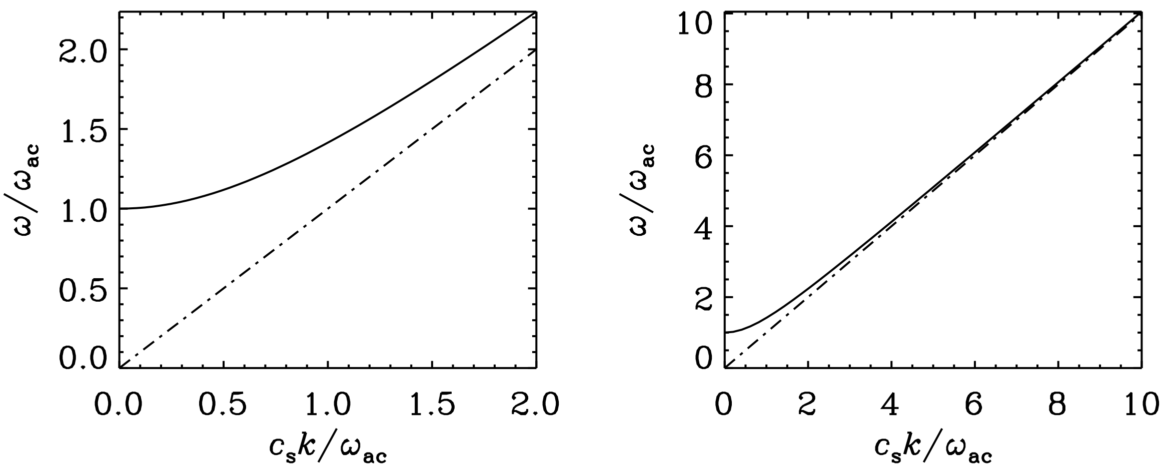

Figure 1: Dispersion relation.