TIME DISCRETISATION: Magnetic fields - integrating factor

We can solve the horizontal part of the diffusion in k-space by an integrating factor.

The advantage of this is that it is exact and forces more diffusion at higher wavenumbers unlike many other schemes (e.g. Crank-Nicolson which has a wavelike attenuation at the highest wavenumbers in its rational polynomial approximation to the exponential).

$$\begin{aligned} \partial_t P &= F + C_k \zeta \left(\nabla_h^2 P + \partial_z^2 P \right), \\ \partial_t P + C_k \zeta k^2 P &= F + C_k \zeta \partial_z^2 P, \end{aligned}$$

where P is $P(k)$ in k-space and

$$F = \frac{\text{(nonlinear)}}{k^2}.$$

So then

$$\begin{aligned} e^{C_k \zeta k^2 t}(\partial_t P + C_k \zeta k^2 P) &= e^{C_k \zeta k^2 t}(F + C_k \zeta \partial_z^2 P), \\ \partial_t (e^{C_k \zeta k^2 t} P) &= e^{C_k \zeta k^2 t}(F + C_k \zeta \partial_z^2 P), \end{aligned}$$

Time discretisation (AB3 for F, Euler for diffusion) gives

$$P^{n+1} E(t^{n+1}) - P^n E(t^n) = c_0 F^n E(t^n) + c_1 F^{n-1} E(t^{n-1}) + c_2 F^{n-2} E(t^{n-2}) + \delta t_0 C_k \zeta \partial_z^2 P^n$$

where $E(t) = e^{C_k \zeta k^2 t}$, so

$$P^{n+1} = E(-\delta t_0) \left(P^n + c_0 F^n + c_1 F^{n-1} E(-\delta t_1) + c_2 F^{n-2} E(-\delta t_1 - \delta t_2) + \delta t_0 C_k \zeta \partial_z^2 P^n \right.$$

where δt_0 is the timestep computed for the next step $n \to n+1$, δt_1 is the timestep for the previous step $n-1 \to n$, and δt_2 is the timestep for the step before that $n-2 \to n-1$.

Disadvantage: have to calculate exponentials – can be expensive