## TIME DISCRETISATION: Magnetic fields - integrating factor

We can solve the horizontal part of the diffusion in $k$-space by an integrating factor.

The advantage of this is that it is exact and forces more diffusion at higher wavenumbers unlike many other schemes (e.g. Crank-Nicolson which has a wavelike attenuation at the highest wavenumbers in its rational polynomial approximation to the exponential).

$$\begin{aligned} \partial_t P &= F + C_k\zeta\left(\nabla_h^2 P + \partial_z^2 P\right), \\ \partial_t P + C_k\zeta k^2 P &= F + C_k\zeta\partial_z^2 P, \end{aligned}$$

where $P$ is $P(k)$ in $k$-space and

$$F = \frac{(\text{nonlinear})}{k^2}.$$

So then

$$\begin{aligned} e^{C_k\zeta k^2 t}(\partial_t P + C_k\zeta k^2 P) &= e^{C_k\zeta k^2 t}(F + C_k\zeta\partial_z^2 P), \\ \partial_t(e^{C_k\zeta k^2 t}P) &= e^{C_k\zeta k^2 t}(F + C_k\zeta\partial_z^2 P), \end{aligned}$$

Time discretisation (AB3 for $F$, Euler for diffusion) gives

$$P^{n+1}E(t^{n+1}) - P^n E(t^n) = c_0 F^n E(t^n) + c_1 F^{n-1}E(t^{n-1}) + c_2 F^{n-2}E(t^{n-2}) + \delta t_0 C_k\zeta\partial_z^2 P^n$$

where $E(t) = e^{C_k\zeta k^2 t}$, so

$$P^{n+1} = E(-\delta t_0)\left(P^n + c_0 F^n + c_1 F^{n-1}E(-\delta t_1) + c_2 F^{n-2}E(-\delta t_1 - \delta t_2) + \delta t_0 C_k\zeta\partial_z^2 P^n\right.$$

where $\delta t_0$ is the timestep computed for the next step $n \to n+1$, $\delta t_1$ is the timestep for the previous step $n-1 \to n$, and $\delta t_2$ is the timestep for the step before that $n-2 \to n-1$.

Disadvantage: have to calculate exponentials – can be expensive