THE ROLE OF SINGULARITIES IN EULER

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Abstract: Possible constraints on how the three–dimensional incompressible Euler equations can have a finite–time singularity from recent numerical calculations and analytic theory are discussed.

1 Introduction

There is now numerical evidence (Kerr 1993) that would support the existence of a singularity of the three-dimensional incompressible Euler equations, or as a minimum gives strong indications of what type of constraints must be satisfied if there is a singularity. But the constraints indicated by these calculations, and additional constraints implied by recent analytic work (Constantin 1994) that is consistent with the numerical calculations, shows that the self–similar form of the collapse to a singularity is not simple. In particular, while simplified models have given us insight into this problem, some new approaches will be necessary to further our understanding of this problem. Some of the new analytic requirements are only hinted at vaguely in the recent review by Constantin or are known by word of mouth, so only how analytic requirements might relate to the numerical calculations will be discussed here. The primary point that is to be emphasized is that from the structure and scaling of the Euler calculations the collapse seems to be divided into two regions, neither of which can collapse in a completely self–similar way.

2 Scaling from Numerical Simulations

To illustrate the structure of the vorticity field a three–dimensional volume rendering of the vorticity squared at a late times for the Euler calculation (Kerr 1993) is shown in Fig. 1. More complete contour plots with some three–dimensional wire frame plots of the same flow have appeared (Kerr 1993), and color versions of this plot are available throught the World-Wide Web (Kerr 1995). Only 1/2 of one of the pair of anti–parallel vortices is shown from a late time when vorticity scaling is consistent with singular behavior. The view is through one of the symmetry planes. Let us divide the vorticity field into two regions. First, there is an bright inner region within which the positions of the peaks of all the strain components behave in a self–similar fashion described earlier (Kerr 1993). In the perspective shown in Fig. 1 the inner region could be characterized as two sheets of vorticity meeting at a point. From a fully three–dimensional perspective the most intense vorticity appears to lie on the surface of a blunt cone, with the tip of the cone being the bright vertex of intersecting vortex sheets in the symmetry plane in Fig. 1. Going out from the bright inner region are lower intensity wings that continue the sheet structure away from the inner region, with some additional complications probably associated with winding of vortex lines. The numerical evidence is that while the vorticity in the outer wings is smaller than that in the inner region, where blowup in the vorticity consistent with the analytic requirements is found, it is also growing in a singular manner in these outer wings. Two regions, that is two length scales, seem to be required if a number of curious features of the Euler calculations are to be explained. The primary evidence given (Kerr 1993) to support the existence of a singularity in the Euler calculations is consistency with the analytic requirement (Beale et al. 1984) that the peak or L^∞ norm of vorticity and related properties blow up as $1/(T-t)^\gamma$, with $\gamma \geq 1$, and probably $\gamma \equiv 1$ by dimensional arguments. In the Euler calculations it is observed in addition that the production of enstrophy, a global quantity, blows up in the same fashion, and by inference the enstrophy itself blows up logarithmically. There is also weak numerical evidence that even the peak velocity blows up as $1/(T-t)^{1/2}$. Both of these additional observations would require some type of blowup outside the inner region collapsing as $L \sim (T-t)$ identified earlier (Kerr 1993).

To understand the different scaling properties of the collapse in the two regions let us start by considering what growth of vortex line length implies about the outer region. If there is blowup in vorticity at any point, this must generate infinite line length. That implies that the vorticity must either be wound up very tightly at some location for there to be a singularity or there must be some direction into which the vortex line escapes. The color plots (Kerr 1995)with vortex lines indicate some winding, but the primary mechanism for the growth of vortex line length seems to be directed outwards from the inner region along the manifold of collapsing vortex sheets. In the direction of vortex line growth there is stretching and growth of the singular outer region that is balanced by collapse in only one direction, perpendicular to the surface of the cone. At any point in the outer regions there does not appear to be much stretching or compression in the third direction, which would follow circles drawn around the axis of the cone of collapsing vorticity, that is around an approximately 45° line drawn up and to the right from the bright vertex on the symmetry plane in Fig. 1. From these observations, it appears that while the collapse of the outer region and its vorticity growth are singular, the curvature of vorticity in this region is not strong. Since the curvature is not strong, the outer region cannot be the source of the singular growth of vortex line length, it only accommodates the growth from the inner region.

If the inner region is the source of the growth of vortex line length, it must have strong curvature of vortex field lines. Strong curvature seems to appear only at the blunt tip of the cone, with sharpness or curvature becoming singular as the singular time is approached. Strong curvature is also a requirement if the inner region is to collapse in a nearly self–similar fashion in all directions, behavior which analysis of the Euler calculations in progress by Brandenburg and Kerr supports. Strong curvature is required due to incompressibility, since locally the strain must always have one stretching, that is expanding not collapsing, direction. This can be accommodated if there is also strong curvature in that direction, that is there must be singular curvature in the direction of the vorticity within a collapsing, singular distance of the position of peak vorticity if there is to be a singularity. Whether the published analytic constraints on curvature if there is to be a singularity (Constantin 1994) are this strong is not yet clear, but such a requirement would rule out some models of the Euler singularity (Bhattacharjee & Wang 1992).

3 Analytic theory

What can be said analytically beyond the published proof (Beale et al. 1984) on the constraint of the $||\omega||_{L^\infty}$ and its correlary (Ponce 1985)? It is claimed that there is a proof that you cannot have exact self-similarity if there is to be a singularity of Euler. To be more precise, the Euler calculations would make it implausible that the restrictive condition necessary for there to be exact self–similarity could exist. If there is to be exact self–similarity the vorticity should obey

$$\omega(x,t) \equiv \frac{1}{T-t}\Omega\Big(\frac{x}{L(t)}\Big)$$

A condition on the length scale that is necessary for there to be a singularity is that

$$\int_0^T \big(\ell_\mu(t)\big)^{-5/2} dt = \infty \quad ,$$

which comes from the use of the Hölder seminorm is. From this it is claimed that if

$$L(t) = L(0)(1-\frac{t}{T})^p$$

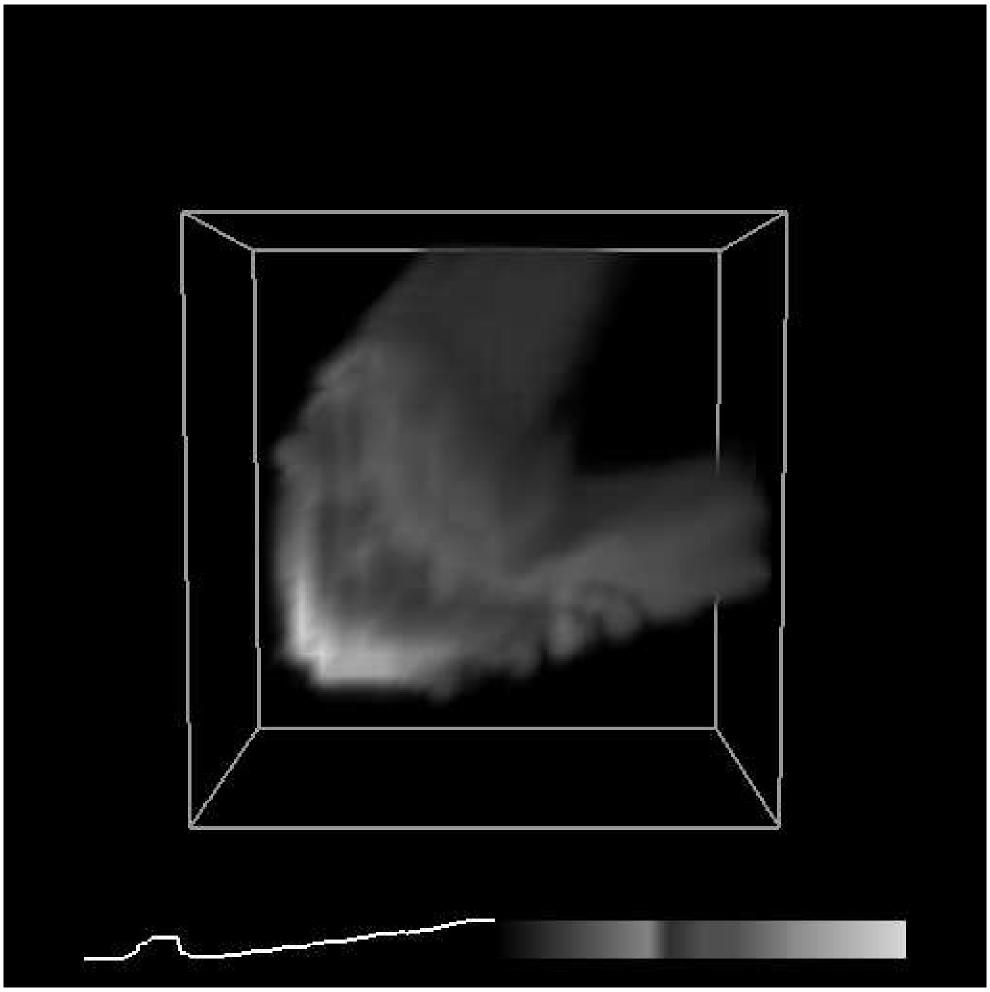

Fig. 1. Volume rendering of Euler calculation (Kerr 1993) at $t = 15$.

then for there to be exact self–similarity and blowup, it is required that

$$p \equiv \frac{2}{5}$$

In the immediate vicinity of the peak vorticity of the Euler calculations the collapse is at least approximately self–similar with $p = 1$. In the outer collapsing region, perhaps $p = 1/2$. Therefore, exact self–similarity cannot be obeyed, but the condition on ℓ_μ necessary for singularity would be obeyed.

What do the analytic results say about the alignment of vorticity and curvature? According to (Constantin 1994) the important quantity is $|\sin\phi|$ where ϕ is the angle between the vorticity at one point x and another point y. This feeds into the formula for strain along the vorticity, which is what stretches the vorticity and causes blowup. $|\sin\phi| = 0$ if the vorticity at points x and y is either parallel or antiparallel. It might be argued that if the vorticity at all points x and y within the inner collapsing region around the peak vorticity is approximately parallel there cannot be a singularity since parallel vorticity can be represented by vortex filaments. For parallel vortex filaments the self–induction approximation can be used and this is known to lead to soliton behavior such as the Hasimoto soliton (Hasimoto 1972), which does not have finite–time singular behavior. Numerical results (Siggia 1985), (Boratav et al. 1992) would suggest that for there to be singular growth the vorticity always rearranges itself so as to be anti–parallel, which is why this type of initial condition has recently dominated numerical studies of Euler singularities and vortex reconnection.

The point is also made (Constantin 1994) that the pressure hessian

$$\pi_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j}$$

plays an important role the dynamics of the direction of vorticity. This is consistent with analysis using the pressure hessian and the second and third invariants of the stress tensor for a model of a restricted Euler equation (Cantwell 1993). For example, for a single vortex filament the second invariant

$$Q = \frac{1}{2}\omega^2 - |e_{ij}|^2$$

is always positive, and ongoing of analysis of data from the Euler calculations by Cheng, Cantwell, and Kerr suggests that somewhere within the inner region around the peak vorticity in Fig 1, the second invariant must be negative $Q < 0$ for there to be singular behavior. This seems to be possible only for the anti–parallel case.

What has been published (Constantin 1994) on how $|\sin\phi|$ might constrain Euler is that if R is defined to be that radius within which the following is obeyed,

$$|\sin\phi(x, x+y)| \leq \frac{|y|}{R}$$

Then apparently by making some reasonable assumptions on an outer lengths scale and a variant of the L^1 norm of vorticity not being singular, it seems that it can be shown that $R \to 0$ is a requirement for blowup. In other words, just outside R, $|\sin\phi|$ becomes very large. But this does not seem as strong as saying that the curvature in the direction of vorticity at the position of the peak vorticity is becoming singular in all directions just outside R. The geometry at the tip of the singular cone suggested by Fig. 1 would suggest that there should be a constraint of this type that states that the curvature should be large in all directions, that is the flow cannot reduce to two–dimensionality in any sense. Such a requirement would rule out recent assertions that axisymmetric flows, or equivalently two–dimensional Boussinesq convection, can have finite–time singularities (Grauer & Sideris 1991), although Moore type singularities in vortex sheets are still a possibility (Caflisch 1993).

If the blowup in enstrophy in Euler is only logarithmic, this would suggest that the L^2 norm of vorticity is some sort of limiting case. For example, Euler might blow up in L^N, $N \geq 2$, but not for $N < 2$. It has been related to me by Majda that there is a new analytic result consistent with this property of the Euler calculations. Enstrophy growing logarithmically also suggests that as the singularity is approached the spectrum will go to k^{-3}, which is supported by the Euler calculations. This brings up a number of points. First, this is not turbulence, which has a spectrum more like $k^{-5/3}$. k^{-3} is also the marginal spectrum between having local and non–local interactions in wavenumber space, with -5/3 being strongly in the regime with only local interactions, that is a local cascade, in wavenumber space. The spectrum approaches k^{-3} not by gradually filling out a -3 spectrum, but starting with very, very steep spectra, the order of k^{-6} when singular behavior begins, then the power law approaches k^{-3}. No exponential regime is ever observed. This growth over all scales suggests very non–local interactions in physical space, which would be consistent with there being two collapsing regions in physical space.

4 Discussion

In searching for answers to the Euler singularity problem a close interplay between the analytic theory and the numerical calculations has proven essential. To date, every known, published or unpublished, constraint upon the existence of a singularity has been met by the best numerical calculations (Kerr 1993). It is therefore important that formal proofs for these theorems appear. The vague descriptions of possible new constraints on the effects of curvature appearing here need to be put in a proper mathematical framework and proofs for these suggestions should be sought. Finally, new scaling analysis of the Euler calculations is needed to provide more insight into what properties might be amenable to analytic treatment.

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