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ON SOME ELECTROMAGNETIC PHENOMENA IN ELECTRICALLY CONDUCTING TURBULENTLY MOVING MATTER, ESPECIALLY IN THE PRESENCE OF CORIOLIS FORCES

There is a lot of problems in astrophysics, in geophysics and also in technical science, which give rise to investigations of electromagnetic fields in electrically conducting turbulently moving matter. During recent years, STEENBECK, KRAUSE, and the author have developed some features of an electrodynamics of mean fields in continuous conducting media showing turbulent motions [1, 2, 4, 6, 7, 9]. After summarizing the basic ideas shown there, some results which are of interest to solar physics, especially for turbulent motions influenced by Coriolis forces are discussed in this paper.

Basic ideas

The starting-point are Maxwell's and the corresponding constitutive equations inclusive Ohm's law, all with the neglections usual in magnetohydrodynamics. We introduce the magnetic flux density B, the magnetic field strength H, the electric field strength E, the electric current density j, and the velocity v of the medium. Both the permeability μ and the electric conductivity σ of the medium are supposed to be constant. The set of equations specified above reads as follows

$$\operatorname{curl} E = -\frac{\partial B}{\partial t}, \qquad \operatorname{curl} H = j, \qquad \operatorname{div} B = 0$$
$$B = \mu H, \qquad j = \sigma\,(E + v \times B). \tag{1}$$

By means of these equations B, H, E, and j may be determined, if v as well as suitable initial and boundary conditions are given. For this end it is useful to reduce (1) to

$$\frac{1}{\mu\sigma}\,\Delta B + \operatorname{curl}\,(v \times B) - \frac{\partial B}{\partial t} = 0, \qquad \operatorname{div} B = 0. \tag{2}$$

After the determination of B from these equations, H, E, and j result immediately by means of (1).

In a turbulently moving medium B, H, E, and j like v show random fluctuations. We use an averaging procedure which yields for each quantity of this type, say F, an averaged or mean quantity $\overline{F}$, and we denote $F - \overline{F}$ by F'. Thus it holds

$$B = \overline{B} + B', \qquad v = \overline{v} + v'. \tag{3}$$

The average may be taken over suitable ranges in space or time or by means of statistics. Only Reynolds' rules and the commutation rule for averaging and derivation or integration are needed for the following.

We regard the determination of $\overline{B}$, $\overline{H}$, $\overline{E}$, and $\overline{j}$ from $\overline{v}$ and some quantities characterizing v' as the central problem in the electrodynamics of the mean fields. Taking the average of (1), one gets

$$\operatorname{curl} \overline{E} = -\frac{\partial \overline{B}}{\partial t}, \qquad \operatorname{curl} \overline{H} = \overline{j}, \qquad \operatorname{div} \overline{B} = 0,$$
$$\overline{B} = \mu \overline{H}, \qquad \overline{j} = \sigma\,(\overline{E} + \overline{v} \times \overline{B} + \overline{v' \times B'}). \tag{4}$$

The formal correspondance of (1) and (4) is disturbed only by the appearance of the term $\overline{v' \times B'}$. It describes an additional electromotive force, effective only for the mean fields. In order to find out $\bar{B}$, $\bar{H}$, $\bar{E}$, and $\bar{J}$, not only $\bar{v}$, but also $\overline{v' \times B'}$ must be known. By combining (2) and (3) follows

$$\frac{1}{\mu\delta} \Delta (\bar{B} + B') + \text{curl} (\bar{v} \times \bar{B} + \bar{v} \times B' + v' \times \bar{B} + v' \times B') - \frac{\partial}{\partial t} (\bar{B} + B') = 0,$$
$$\text{div} (\bar{B} + B') = 0. \tag{5}$$

Principally, by means of these equations B' may be represented by $\bar{B}$, $\bar{v}$, and v'. Consequently, it is allowed to consider even $\overline{v' \times B'}$ as a functional of $\bar{B}$, $\bar{v}$, and v'. If $\overline{v' \times B'}$ is replaced by an expression of that kind, the equations (4) make possible the determination of $\bar{B}$, $\bar{H}$, $\bar{E}$, and $\bar{J}$ from $\bar{v}$ and quantities depending only on v'.

Of course, the actual determination of $\overline{v' \times B'}$ is very difficult. Here we sketch only a few ideas allowing some insight into the structure of $\overline{v' \times B'}$ and give the general result.

It is to be expected that fluctuating quantities as v' and B' at a given point in space and time show not correlation with any quantities at any points far away from the given one. Consequently, $\overline{v' \times B'}$ for a specific point depends only on the behaviour of $\bar{B}$, $\bar{v}$, and v' in a confined surrounding of it. Furthermore, it may be easily concluded from (5) that $\overline{v' \times B'}$ regarded as functional of $\bar{B}$, $\bar{v}$, and v' is linear with respect to $\bar{B}$. For simplification let us assume that $\bar{B}$ varies only weakly with the position and may be considered as constant in time. More precisely, in the mentioned surrounding of the point for which $\overline{v' \times B'}$ is considered the behaviour of $\bar{B}$ is supposed to be determined only by $\bar{B}$ itself and its first spatial derivatives in this point. In terms of the tensor calculus, therefore we have

$$(\overline{v' \times B'})_i = a_{ij} \bar{B}_j + b_{ijk} \frac{\partial \bar{B}_j}{\partial x_k}. \tag{6}$$

Obviously, the tensors a_{ij} and b_{ijk} are averaged quantities depending only on $\bar{v}$ and v'. As we shall point out in the following, in some simple cases a_{ij} and b_{ijk}, except for some scalar factors, may be found easily from the symmetry properties of $\bar{v}$ and v'.

For the complete determination of $\overline{v' \times B'}$ relations derived from (5) without fundamental restrictions are available [4, 6, 7]. We have

$$(\overline{v' \times B'})_i = \iiint\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} K_{ij} (x, t; \xi, \tau) \bar{B}_j (\xi, \tau) \, d^3\xi \, d\tau, \tag{7}$$

where K_{ij} contains a tensorial Green's function depending on $\bar{v}$ and correlation tensors for v'. In a first approximation for small v' holds

$$K_{ij}(x, t; \xi, \tau) = - \varepsilon_{ikl} \varepsilon_{mnp} \varepsilon_{pqj} \frac{\partial G_{lm}(x, t; \xi, \tau)}{\partial \xi_n} \overline{v'_k(x, t) \; v'_q(\xi, \tau)}. \tag{8}$$

G_{lm} denotes the tensorial Green's function mentioned. In higher approximations besides the correlation tensor $\overline{v'_k v'_q}$ of the second rank even such expressions of higher ranks occur.

Turbulence without Coriolis forces

As the simplest example we consider a homogeneous isotropic turbulence. In this case $\bar{v}$ vanishes and all averaged quantities containing v' are invariant against arbitrary translations of the v' field and against arbitrary rotations of them around arbitrary axes. By reason of this we conclude that a_{ij} and b_{ijk} are constant in space and may be represented by $\alpha\delta_{ij}$ and $\beta\varepsilon_{ijk}$. Inserting this in (6), α is found to be a pseudoscalar but β a scalar. For a homogeneous isotropic turbulence it is to be expected that all averaged quantities containing v' are even invariant against reflexions of the v' field on arbitrary planes. (Several authors include this reflexion symmetry into the definition of isotropy.) Thus it is impossible to derive a pseudoscalar from those quantities, and therefore α vanishes. So we have

$$\overline{v' \times B'} = - \beta \text{ curl } \overline{B}. \tag{9}$$

With regard to $\overline{v}$ being zero, the last equation of (4) may be rewritten to

$$\overline{J} = \sigma_T \overline{E}, \qquad \sigma_T = \sigma/(1 + \mu\sigma\beta). \tag{10}$$

For the mean fields in a turbulently moving medium therefore a 'turbulent' conductivity σ_T different from the original σ must be taken into account.

Using (7), (8), and a correlation tensor derived from the mentioned suppositions about the v' field, calculations were carried out providing σ_T in a first approximation for small v' [4, 7]. As pointed out in this connection, in the convection zone of the sun the condition $\mu\sigma\lambda_K^2 \gg \tau_K$ is satisfied, where λ_K and τ_K are correlation length and correlation time in the usual sense. Under this condition and the further one $v'^2 \ll (\lambda_K/\tau_K)^2$ limiting the first approximation holds

$$\sigma_T = \sigma/(1 + \tfrac{1}{3}\mu\sigma\overline{v'^2}\tau_K) \;^{*)}. \tag{11}$$

Thus $\sigma_T \approx 10^{-4}\,\sigma$ seems to be possible in the convection zone.

Next we deal with a turbulence deviating from a homogeneous isotropic one only due to a gradient of either intensity or correlation length. In the following, g equals either $(1/\overline{v'^2})$ grad $\overline{v'^2}$ or $(1/\lambda_K^2)$ grad λ_K^2. In order to determine a_{ij} and b_{ijk} at a given point in space and time, we consider the v' field in an adequate surrounding of it and suppose g to be constant there. Restricting ourselves to linearity with respect to g, for a_{ij} results a sum of $\alpha\delta_{ij}$ and $\gamma\varepsilon_{ijl}\,g_l$, for b_{ijk} a sum of $\beta\varepsilon_{ijk}$, $\beta_1\delta_{ij}g_k$, $\beta_2\delta_{ik}g_j$, and $\beta_3\delta_{jk}g_i$. Since g is a polar vector, according to (6) not only α, but also β_1 and β_2 are pseudoscalars, whereas β and γ are scalars; β_3 is insignificant because of $\partial\overline{B}_i/\partial x_i = 0$. In compliance with the supposed linearity with respect to g, these quantities cannot depend on g. Consequently, for the determination of them the turbulence must be regarded as a homogeneous isotropic one. Accepting again the latter shows reflexion symmetry in the sense explained above, it may be concluded that α as well as β_1 and β_2 are zero. As a result, we have

$$\overline{v' \times B'} = - \beta \text{ curl } \overline{B} - \gamma\, g \times \overline{B} \tag{12}$$

and furthermore

$$\overline{J} = \sigma_T\,(\overline{E} - \gamma\, g \times \overline{B}). \tag{13}$$

The gradient causes mean currents flowing perpendicular to both itself and the mean flux density.

According to g being $(1/\overline{v'^2})$ grad $\overline{v'^2}$ or $(1/\lambda_K^2)$ grad λ_K^2, we replace γ by γ_v or γ_λ. Calculations were carried out for γ_v [4, 7] and for γ_λ in the same way as done for σ_T. In the case of $\mu\sigma\lambda_K^2 \gg \tau_K$ and $\overline{v'^2} \ll (\lambda_K/\tau_K)^2$ holds

$$\gamma_v = \tfrac{1}{6}\,\overline{v'^2}\tau_K, \qquad \gamma_\lambda = \tfrac{5}{6}\,\frac{\overline{v'^2}\tau_K^2}{\mu\sigma\lambda_K^2} \;^{*)}. \tag{14}$$

Under certain circumstances, the currents accompanying the gradient make allowance for the introduction of a 'turbulent' permeability different from the usual one [4, 7].

Turbulence influenced by Coriolis forces

Contrary to the cases treated before, we now take into account a mean motion of the medium, that is to say a non vanishing $\bar{v}$. As a simple mean motion we consider a rotation around an axis characterized by an angular velocity ω. Of course, then the v' field is influenced by the Coriolis and centrifugal forces depending on ω. We restrict ourselves to linearity with respect to ω. Thus only Coriolis forces are of interest. For simplification we suppose that not only $\bar{v}$ but also all the averaged quantities derivable from v' and even $\bar{B}$ are symmetrical with respect to the axis of rotation. Thus $\overline{v' \times B'}$ regarded as functional of $\bar{B}$, $\bar{v}$, and v' depends no longer on $\bar{v}$ [9]. Therefore a_{ij} and b_{ijk} are determined by v' alone. For the investigation of them at a given point in space and time like g used above also ω is taken as constant.

At first the turbulence which is overlaying the mean motion is assumed to differ from a homogeneous isotropic one only due to the influence of Coriolis forces. In this case we may use the conception developed in connection with the gradient in the properties of the v' field. However, the polar vector g must be replace by the axial vector ω. Therefore α and γ are pseudoscalars and must be zero, whereas β, β_1, and β_2 are scalars. It is useful to change the signs of β_1 and β_2. Thus, instead of (12) we get

$$\bar{j} = \sigma_T\,(\bar{E} + \bar{v} \times \bar{B} - \beta_1\,(\omega\,\mathrm{grad})\,\bar{B} - \beta_2\,\omega\,\mathrm{grad}\,\bar{B}) \qquad (15)$$

where $(\omega\,\mathrm{grad}\,\bar{B})_i = \omega_j\,\partial \bar{B}_j/\partial x_i$. With the aid of (1) it may be rewritten into

$$\bar{j} = \sigma_T\,(\bar{E} + \bar{v} \times \bar{B} + \mu\beta_1\;\omega \times \bar{j} - (\beta_1 + \beta_2)\,\omega\,\mathrm{grad}\;\bar{B}). \qquad (16)$$

The most interesting feature of this result is the occurence of the term $\mu\beta_1\,\omega \times \bar{j}$ suggesting the Hall effect. As may be concluded from this comparison, this term generally describes a tendency to distort the direction of the mean current and gives rise to the introduction of a tensorial conductivity depending on ω. The term $-(\beta_1 + \beta_2)\omega\,\mathrm{grad}\,\bar{B}$ is of minor importance. So far as ω, β_1, and β_2 are constant, it is equal to $-(\beta_1 + \beta_2)$ grad $(\omega\bar{B})$ and may be compensated always be a part of $\bar{E}$ resulting from space charges.

Though general formulae for β_1 and β_2 are available [9], detailed calculations were carried out only for an incompressible medium, that is to say div $v' = 0$. If $\mu\sigma\lambda_K^2 \gg \tau_K$ and $\overline{v'^2} \ll (\lambda_K/\tau_K)^2$ we get

$$\beta_1 = \frac{8}{45}\,\frac{\overline{v'^2}\,\lambda_K^2\,\tau_K}{\mu\,\sigma\,\nu^2}, \qquad \beta_2 = -\frac{2}{9}\,\frac{\overline{v'^2}\,\lambda_K^2\,\tau_K}{\nu}\,f\,\left(\frac{\lambda_K^2}{\nu\tau_K}\right) \qquad (17)$$

where ν denotes the kinematic viscosity of the medium and f is given in the diagram. It was pointed out that the modification of Ohm's law discussed here makes possible the maintenance of electromagnetic fields by dynamo action [10].

Finally, a turbulence is considered which deviates from a homogeneous isotropic one firstly due to a gradient in intensity of correlation length and secondly because of Coriolis forces. Again restricting ourselves to linearity with respect to both g and ω in an analogous manner as before we get

$$\begin{aligned}\bar{j} = \sigma_T\,(\bar{E} + \bar{v} \times \bar{B} - \gamma\,g \times \bar{B} - \beta_1\,(\omega\,\mathrm{grad})\,\bar{B} - \beta_2\,\omega\,\mathrm{grad}\,\bar{B} \\ - \alpha_1\,(g\omega)\,\bar{B} - \alpha_2\,(\omega\bar{B})g - \alpha_3\,(g\bar{B})\omega).\end{aligned} \qquad (18)$$

The most remarkable term in this relation is $-\alpha_1\ (g\omega)\ \bar{B}$ which describes an electromotive force parallel or antiparallel to the mean magnetic flux. In all the cases considered before, the turbulence shows reflexion symmetry at least with respect to some special planes. That is why terms of that kind did not occur. This reflexion symmetry is violated here.

Like γ_v and γ_λ we introduce α_{1v}, $\alpha_{1\lambda}$, and so on. Several results for α_{1v}, α_{2v}, and α_{3v} were given previously [2, 6], however, some details of them have to be corrected. Again we restrict ourselves do div v' = 0 as well as $\mu\delta\lambda_K^2 \gg \tau_K$ and $\overline{v'^2} \ll (\lambda_K/\tau_K)^2$. Thus we have

$$\alpha_{1v} = \alpha_{1\lambda} = \frac{4}{45}\,\frac{\overline{v'^2}\,\lambda_K^2\,\tau_K}{\nu} \qquad (19)$$

$$\alpha_{2v} = \alpha_{2\lambda} = -\frac{2}{15}\,\frac{\overline{v'^2}\,\lambda_K^2\,\tau_K}{\nu}$$

$$\alpha_{3v} = \alpha_{3\lambda} = -\frac{2}{15}\,\frac{\overline{v'^2}\,\lambda_K^2\,\tau_K}{\nu}\left(1 + \frac{5}{3}\, f\left(\frac{\lambda_K^2}{\nu\tau_K}\right)\right).$$

Taking into account no other electromotive force caused by turbulence but the discussed one parallel or antiparallel to the mean magnetic flux, also a maintenance of electromagnetic fields by dynamo action is possible [2]. A series of dynamo models were treated on this basis [3, 5]. Especially a model for alternating fields allowing an interpretation of the magnetic phenomena at the sun was investigated in detail [8].

*) The numerical values in (11) and (14) are in agreement with [4], but they differ a little from the values given in [1]. The reason for it consists in different assumption on the shape of correlation functions.

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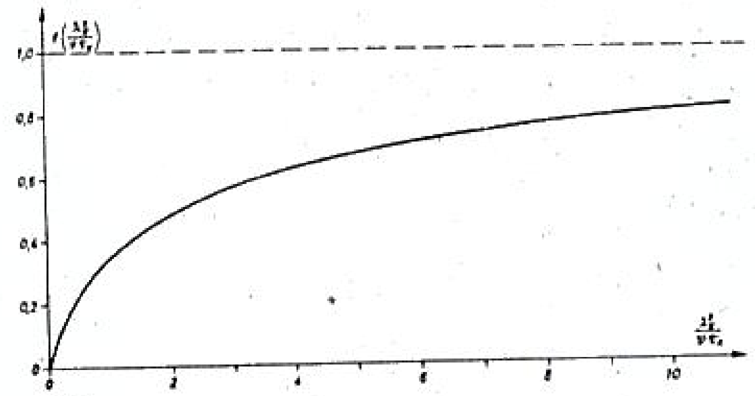

Fig.1