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Hydrodynamic Green's functions for atmospheric oscillations

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Summary. A Green's function formalism for isothermal atmospheres is employed to solve an initial value problem in the linear and adiabatic approximation taking compressible effects fully into account. This method is applied to Brunt-Väisälä oscillations of axisymmetric bubbles. The solution for the velocity, entropy and pressure field is given as a double integral, which may be evaluated either approximately or rigorously by quadrature. A localized initial disturbance (e.g. in entropy) can excite different kinds of gravity waves, depending on the size of the disturbance and on the strength of coupling between gravity and pressure modes. Comparison is made with the anelastic approximation, which turns out to give correct results only if the size of the initial disturbance is small enough (smaller than about ten pressure scale heights), or if the ratio of the specific heats γ is close to unity, i.e. if the atmosphere is close to marginal convective stability. The anelastic approximation becomes invalid, if the group velocity of gravity waves gets close to the speed of sound.

Key words: analytical methods: Green's functions – hydrodynamics – anelastic approximation – oscillations of the Sun, g-modes

1. Introduction

The theory of linearized flows with initial and boundary conditions modeling a bubble has been of interest in different circumstances. Meyer and Schmidt (1967) and Stix (1970) calculated wave generation due to granulation assuming a rising and falling bubble as the lower boundary condition. They expanded the solution in terms of eigenfunctions and found several properties observed in the solar granulation, e.g. the variation with height of amplitude, frequency, power-spectrum and phase between temperature and velocity fluctuations.

Another application of bubble-like flow has been the possibility of testing hydrodynamic computer codes starting from an undisturbed state. Stefanik et al. (1984), Gigas and Steffen (1984) and Schmitz (1986) considered the motion of a hot bubble in an isothermal atmosphere and computed Brunt-Väisälä oscillations with a hydro-code based on the method of characteristics (Stefanik et al., 1984). They compared period and maximum velocity with the quantities obtained from the theory of harmonic oscillations. However, damping and a type of variation of the velocity field was also found, which is not predicted by simple theory. Apart from analytic solutions of standing waves in a cylinder (Stefanik et al., 1984) no other solutions are available for quantitative comparison.

The goal of the present paper is to study in more detail a laminar flow resulting from a bubble, undergoing Brunt-Väisälä oscillations. In particular we shall consider effects due to large-scale disturbances, which have not been considered in the papers mentioned above. Furthermore we shall discuss the anelastic approximation (Gough, 1969), which has been widely used in astrophysics to model convection (e.g. Toomre et al., 1976; Nordlund, 1982).

We make use of the method of Green's functions to compute the response of an atmosphere to arbitrary disturbances in velocity, entropy and pressure. This method can be regarded as a generalisation of the modal analysis treated by Meyer and Schmidt (1967). In astrophysics the Green's function formalism has been employed in several cases. For example the response of a star's radius and entropy due to mass and energy transfer (Hazlehurst et al., 1977) or due to arbitrary spherical disturbances of the hydrostatic stratification (Däppen, 1983) can be given by Green's functions. Aizenman and Perdang (1976) used Green's functions to study the secular stability of a star considering perturbations in the chemical abundances. Recently Bräuer and Rädler (1986, 1987) reported on Green's functions for the equation of induction, which is important in solar dynamo theory.

This paper is arranged as follows: in the second section we shall derive a hermitian differential operator that describes an adiabatic flow in a plane-parallel isothermal atmosphere. In Sect. 3 this operator will be inverted by means of Fourier transformation to give a Green's function tensor. In the following section the inverse Fourier transformation for an axisymmetric initial condition (e.g. a bubble) is carried out. In Sect. 5 we discuss different properties of g-modes and also the range of validity of the anelastic approximation. An explicit, but approximate solution for small bubbles is given in Sect. 6 and the results are discussed in the last section.

2. The basic equations

We wish to study gravitational and acoustic waves in an atmospheric layer, which may be approximated as being isothermal. We neglect rotation and magnetic fields, which would modify the flow for longer timescales than considered here. For example, if we assume a magnetic field of 300 Gauss in the solar atmosphere an Alfvén wave would need about ten Brunt-Väisälä periods to travel one pressure scale height. This is, of course, no

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longer the case in sunspots. Following Stein and Leibacher (1974), we adopt the inviscid and adiabatic equations, because the dissipative effects do not introduce any new modes. At the bottom of the solar photosphere the radiative cooling time is about one hour, but decreases rapidly towards upper layers (Noyes and Leighton, 1963). Our analysis can therefore only be applied to regions not too close to the surface. Denoting now by $D/Dt = \partial/\partial t + \boldsymbol{v}.\nabla$ the advective derivative we have:

$$\rho \frac{D\boldsymbol{v}}{Dt} + \nabla p - \rho \boldsymbol{g} = 0 \quad (1)$$

$$\frac{Ds}{Dt} = 0 \quad (2)$$

$$\frac{D\rho}{Dt} + \rho \nabla . \boldsymbol{v} = 0 \quad (3)$$

where s is the specific entropy, which is related to the other thermodynamic quantities pressure p and density ρ by:

$$\frac{Ds}{Dt} = c_v \frac{D \log p}{Dt} - c_p \frac{D \log \rho}{Dt} \quad (4)$$

where c_v and c_p are the specific heats at constant density and pressure, respectively, which we assume as being constant. Instead of Eq. (3) we shall use a similar one for the pressure obtained by combining Eqs. (2)–(4):

$$\frac{Dp}{Dt} + \gamma p \nabla . \boldsymbol{v} = 0 \quad (5)$$

Here γ is the ratio c_p/c_v. Equation (4) is used to eliminate the density ρ in Eq. (1). The five unknown quantities are v_x, v_y, v_z, s, and p. We now consider motions with sufficiently small amplitudes and shall therefore linearize Eqs. (1), (2), (4), and (5) about their equilibrium values which correspond to a static isothermal atmosphere:

$$P^{(0)}(z) = P^{(0)}(0) \exp(-z/H_p) \quad (6)$$

$$\gamma P^{(0)}(z)/\rho^{(0)}(z) = c^2 = \text{const.} \quad (7)$$

$$ds^{(0)}/dz = c_p(\gamma - 1)/\gamma H_p = \text{const.} \quad (8)$$

c denotes the speed of sound and $H_p = c^2/\gamma g$ the pressure scale height. The equations for the deviations, denoted by superscript$^{(1)}$, are derived to be:

$$\frac{\partial \boldsymbol{v}^{(1)}}{\partial t} - \boldsymbol{e}_z g \left(\frac{s^{(1)}}{c_p} - \frac{p^{(1)}}{\gamma p^{(0)}} \right) + \frac{1}{\rho^{(0)}} \nabla p^{(1)} = 0 \quad (9)$$

$$\frac{\partial s^{(1)}}{\partial t} + v_z^{(1)} \frac{ds^{(0)}}{dz} = 0 \quad (10)$$

$$\frac{\partial p^{(1)}}{\partial t} + v_z^{(1)} \frac{dp^{(0)}}{dz} + \gamma p^{(0)} \nabla . \boldsymbol{v}^{(1)} = 0 \quad (11)$$

It is possible to organize these five equations in matrix form such that the differential operator matrix becomes hermitian. To this end we now introduce dimensionless quantities, indicated by a prime, as follows:

$$\boldsymbol{x}' = \boldsymbol{x}/H_0 \quad \text{with} \quad H_0 = \gamma H_p/(1 - \gamma/2) \quad (12)$$

$$t' = t/T_0 \quad \text{with} \quad T_0 = H_0/c \quad (13)$$

$$\boldsymbol{v}' = \boldsymbol{v}^{(1)} \exp(-z/2H_p)/(ic) \quad (14)$$

$$s' = s^{(1)} \exp(-z/2H_p)/[ic_p\sqrt{\gamma - 1}] \quad (15)$$

$$p' = p^{(1)} \exp(+z/2H_p)/[i\gamma p^{(0)}(0)] \quad (16)$$

The singularities in this transformation at $\gamma = 1$ and $\gamma = 2$ are not physical and could be removed, if necessary. In the following we shall, however, consider only $1 < \gamma < 2$. The imaginary unit i is introduced for reasons of appearance to obtain a hermitian and not an antihermitian differential operator later on. We define hermiticity here using a scalar product with integration over space *and* time.

We insert now Eqs. (12)–(16) into (9)–(11) and order in matrix form:

$$L_{ij} q_j = 0 \quad i, j = 1, \ldots, 5 \quad (17)$$

where q_j is a column vector and L_{ij} a hermitian differential operator. Explicitly Eq. (17) reads:

$$\begin{pmatrix} i\partial_t & 0 & 0 & 0 & i\partial_x \\ 0 & i\partial_t & 0 & 0 & i\partial_y \\ 0 & 0 & i\partial_t & -i\omega_0 & i + i\partial_z \\ 0 & 0 & i\omega_0 & i\partial_t & 0 \\ i\partial_x & i\partial_y & -i + i\partial_z & 0 & i\partial_t \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \\ v'_z \\ s' \\ p' \end{pmatrix} = 0 \quad (18)$$

ω_0 is a dimensionless Brunt-Väisälä frequency, defined as:

$$\omega_0 = \sqrt{\gamma - 1}/(1 - \gamma/2) \quad (19)$$

Note that L_{ij} is not any longer hermitian if ω_0 becomes imaginary, i.e. $\gamma < 1$. This would correspond to a convectively unstable atmosphere, which will not be considered here.

A hermitian differential operator similar to that in Eq. (18) was considered by Haken (1983) for the Rayleigh-Bénard convection using the Boussinesq approximation.

We shall now discuss the boundary conditions. We shall here restrict ourselves to the case of closed boundaries, i.e.

$$v'_z = 0 \text{ at } z' = \pm L_z \quad (20)$$

at an upper and lower boundary $\pm L_z$. This boundary condition is also consistent with hermiticity. We wish to study the initial value problem, starting from an undisturbed state, and note that a disturbance always needs some time to arrive at the boundaries. We shall therefore take the attitude of assuming the boundaries so far away from the center of the disturbance that its behaviour is independent of the particular choice of the boundary conditions, i.e.

$$L_z \gg \sqrt{\omega_0^2 + 1}\, t \quad (21)$$

The presence of the boundaries, however, is necessary to keep the velocity and pressure disturbances finite and small enough for longer times.

3. The Green's-function tensor

The differential equations (17) may be solved in terms of Green's functions $G_{ij}(t, x)$:

$$q_i(t, \boldsymbol{x}) = \int_{-\infty}^{\infty} d^3x' \, G_{ij}(t, \boldsymbol{x} - \boldsymbol{x}') q_{0j}(\boldsymbol{x}') \quad (22)$$

where $q_{0j}(\boldsymbol{x})$ is the initial condition. The Green's functions have to satisfy the boundary conditions (see discussion in the previous section) and:

$$L_{ij}(t, \boldsymbol{x})\, G_{jk}(t, \boldsymbol{x}) = \delta_{ik}\,\delta(t)\,\delta^3(\boldsymbol{x}) \tag{23}$$

Even in the case of finite boundary conditions the range of integration in Eq. (22) may be taken over the entire space, provided that $G_{ij}(t, \boldsymbol{x})$ vanish outside the domain.

As shown by Bräuer and Rädler (1986) such Green's functions solve at the same time also the related inhomogeneous equations:

$$L_{ij} q_j = Q_i \qquad i, j = 1, \ldots, 5 \tag{24}$$

via:

$$q_i(t, \boldsymbol{x}) = \int_{-\infty}^{\infty} d^3x'\, G_{ij}(t, \boldsymbol{x} - \boldsymbol{x}')\, q_{0j}(\boldsymbol{x}') + \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d^3x'\, G_{ij}(t - t', \boldsymbol{x} - \boldsymbol{x}')\, Q_j(t', \boldsymbol{x}') \tag{25}$$

Fourier transformation is now introduced for q_i and G_{ij}:

$$q_i(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3x\, e^{-i(\boldsymbol{kx} - \omega t)}\, q_i(t, \boldsymbol{x}) \tag{26}$$

$$q_i(t, \boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{i(\boldsymbol{kx} - \omega t)}\, q_i(\omega, \boldsymbol{k}) \tag{27}$$

Corresponding formulae hold also for G_{ij}. Equation (23) can be solved by substituting $\partial/\partial x \to i k_x$ etc. and inverting the resulting algebraic matrix:

$$G_{ij}(\omega, \boldsymbol{k}) = L_{ij}^{-1}(\omega, \boldsymbol{k}) \tag{28}$$

In Fourier space the convolution integral in Eq. (22) is replaced by a common product:

$$q_i(\omega, \boldsymbol{k}) = G_{ij}(\omega, \boldsymbol{k})\, q_{0j}(\boldsymbol{k}) \tag{29}$$

where $q_{0j}(\boldsymbol{k})$ is the Fourier transformation of the initial condition $q_{0j}(\boldsymbol{x})$.

It is convenient to split the hermitian Green's tensor into two parts as follows:

$$G_{ij}(\omega, \boldsymbol{k}) = \frac{\omega}{\det L(\omega, \boldsymbol{k})}\left[\omega G_{ij}^{(c)}(\omega, \boldsymbol{k}) + G_{ij}^{(s)}(\omega, \boldsymbol{k})\right] \tag{30}$$

The determinant of L_{ij} entering into this equation reads:

$$\begin{aligned} \det L &= \omega[(\omega^2 - k_r^2)(\omega^2 - \omega_0^2) - \omega^2(k_z^2 + 1)] \\ &= \omega[\omega^4 - \omega^2(k^2 + \omega_0^2 + 1) + \omega_0^2 k_r^2] \end{aligned} \tag{31}$$

with:

$$k_r^2 = k_x^2 + k_y^2 \quad \text{and} \quad k^2 = k_x^2 + k_y^2 + k_z^2 \tag{32}$$

The zeros of Eq. (31) are the dimensionless solutions of the well known dispersion relation for isothermal atmospheres (e.g. Moore and Spiegel, 1964). In physical dimensions ω_0/T_0 is the Brunt-Väisälä frequency and $\sqrt{(\omega_0^2 + 1)}/T_0$ is the Lamb cutoff frequency.

Employing some simple algebra, we find from Eqs. (28) and (30) the following two matrices $G_{ij}^{(c)}$ and $G_{ij}^{(s)}$.

$$G_{ij}^{(c)}(\omega, \boldsymbol{k}) = \begin{pmatrix} (\omega^2 - k_y^2)(1 - \omega_0^2/\omega^2) - k_z^2 - 1 & k_x k_y (1 - \omega_0^2/\omega^2) & k_x(k_z + i) & 0 & 0 \\ k_x k_y (1 - \omega_0^2/\omega^2) & (\omega^2 - k_x^2)(1 - \omega_0^2/\omega^2) - k_z^2 - 1 & k_y(k_z + i) & 0 & 0 \\ k_x(k_z - i) & k_y(k_z - i) & \omega^2 - k_r^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 - k^2 - 1 & -i\omega_0(k_z - i) \\ 0 & 0 & 0 & i\omega_0(k_z + i) & \omega^2 - \omega_0^2 \end{pmatrix} \tag{33}$$

and:

$$G_{ij}^{(s)}(\omega, \boldsymbol{k}) = \begin{pmatrix} 0 & 0 & 0 & i\omega_0 k_x(k_z + i) & k_x(\omega^2 - \omega_0^2) \\ 0 & 0 & 0 & i\omega_0 k_y(k_z + i) & k_y(\omega^2 - \omega_0^2) \\ 0 & 0 & 0 & i\omega_0(\omega^2 - k_r^2) & (k_z - i)\omega^2 \\ -i\omega_0 k_x(k_z - i) & -i\omega_0 k_y(k_z - i) & -i\omega_0(\omega^2 - k_r^2) & 0 & 0 \\ k_x(\omega^2 - \omega_0^2) & k_y(\omega^2 - \omega_0^2) & (k_z + i)\omega^2 & 0 & 0 \end{pmatrix} \tag{34}$$

4. The inverse Fourier transformation

The problem of finding the solution of Eq. (17) has been replaced by the one of evaluating the Fourier integrals in Eq. (27). The ω-integral may easily be found using the law of residuals (e.g. Schlögl, 1956), because $G_{ij}(\omega, \boldsymbol{k})$ has singularities $\omega_p(\boldsymbol{k})$ and $\omega_g(\boldsymbol{k})$ at the zeros of $\det L_{ij}$:

$$\omega_{p/g}^2 = \tfrac{1}{2}(k^2 + \omega_0^2 + 1)\{1 \pm [1 - 4\omega_0^2 k_r^2/(k^2 + \omega_0^2 + 1)^2]^{1/2}\} \tag{35}$$

We are interested in the retarded Green's functions and therefore we have to shift the singularities into the lower half of the complex ω-plane:

$$\omega \to \omega + i0 \tag{36}$$

Physically this corresponds to introducing a small positive damping into the basic equations.

Carrying out this ω-integration we end up with a new matrix $G_{ij}(t, \boldsymbol{k})$, which we split again into two parts, a p-mode and a g-mode response:

$$G_{ij}(t, \boldsymbol{k}) = G_{ij}^{(p)}(t, \boldsymbol{k}) + G_{ij}^{(g)}(t, \boldsymbol{k}) \tag{37}$$

Both components can be written by means of the matrices $G^{(c)}$ and $G^{(s)}$ (Eqs. 33 and 34) using as ω-argument the frequencies ω_p and ω_g, which are functions of $\boldsymbol{k}$ (Eq. 35):

$$\begin{aligned} G_{ij}^{(p/g)}(t, \boldsymbol{k}) = \pm \frac{\theta(t)}{\omega_p^2 - \omega_g^2}&[\cos(\omega_{p/g} t)\, G_{ij}^{(c)}(\omega_{p/g}, \boldsymbol{k}) \\ &+ \sin(\omega_{p/g} t)\, G_{ij}^{(s)}(\omega_{p/g}, \boldsymbol{k})] \end{aligned} \tag{38}$$

Here $\theta(t)$ denotes the Heaviside step function. From Eq. (38) we see that the components of the tensor $G^{(c)}$ and $G^{(s)}$ differ in phase by $\pi/2$. This leads to the property that the entropy and pressure advance the velocity by a quarter of a period in an oscillatory adiabatic and inviscid flow (Meyer and Schmidt, 1967). In the following we shall study some properties of the Green's functions using special initial conditions.

Stefanik et al. (1984) computed numerically the response of an isothermal atmosphere to an enhanced temperature at gridpoints around the centre of their computational domain. This resembles the liberation of a hot bubble undergoing subsequently Brunt-Väisälä oscillations. We take a similar approach and choose as an initial condition an axisymmetric distribution of the entropy as follows:

$$q_{0j}(r,z) = \delta_{j4}\exp(-r^2/R_r^2 - z^2/R_z^2) \tag{39}$$

with

$$r^2 = x^2 + y^2 \tag{40}$$

and r the distance from the vertical z-axis. This gaussian distribution of entropy corresponds to a bubble-like disturbance with an unsharp boundary and an elliptical cross-section. The Fourier transformation of this initial condition reads:

$$q_{0j}(k_r,k_z) = \delta_{j4}\pi^{3/2}R_r^2 R_z \exp(-k_r^2 R_r^2/4 - k_z^2 R_z^2/4) \tag{41}$$

The Fourier integrals for the solution of $q_i(t,r,z)$ may now be written in cylindrical co-ordinates and the integration with respect to the azimuthal angle can be replaced by the integral representation of the Bessel functions. The remaining double integral reads:

$$\begin{pmatrix} v_r' \\ v_z' \\ s' \\ p' \end{pmatrix} = \frac{R_r^2 R_z}{2\sqrt{\pi}}\int_0^\infty dk_r \int_0^\infty dk_z \frac{\exp(-k_r^2 R_r^2/4 - k_z^2 R_z^2/4)}{\omega_p^2-\omega_g^2} \times$$

$$\begin{pmatrix} -k_r^2 J_1(k_r r)(\cos k_z z + k_z \sin k_z z)\left[\frac{1}{\omega_p}\sin\omega_p t - \frac{1}{\omega_g}\sin\omega_g t\right]\omega_0 \\ k_r J_0(k_r r)\cos k_z z\,[(\omega_p^2-k_r^2)\frac{1}{\omega_p}\sin\omega_p t - (\omega_g^2-k_r^2)\frac{1}{\omega_g}\sin\omega_g t]\omega_0 \\ k_r J_0(k_r r)\cos k_z z\,[(\omega_p^2-k^2-1)\cos\omega_p t - (\omega_g^2-k^2-1)\cos\omega_g t] \\ -k_r J_0(k_r r)(\cos k_z z + k_z\sin k_z z)[\cos\omega_p t - \cos\omega_g t]\omega_0 \end{pmatrix} \tag{42}$$

This result could also have been obtained directly by means of Hankel transformation (see e.g. Sneddon, 1955).

Let us now return to the question of the boundary conditions. The closed boundary conditions Eq. (20) can be fulfilled, if the k_z-integral in Eq. (42) is replaced by a sum over discrete k_z-values:

$$k_z^{(n)} = \pi(n+1/2)/L_z \qquad n = 1, 2, \ldots \tag{43}$$

This procedure is the usual expansion in terms of eigenfunctions with the eigenvalues $\omega_{p/g}(\boldsymbol{k})$ (see Meyer and Schmidt, 1967). The initial condition has, of course, also to be modified such that Eq. (39) is replaced by a Fourier sequence over Eq. (41) with the discrete k_z-values given in Eq. (43). This is because the closed boundary condition requires automatically that the entropy disturbance s' vanish at $z = \pm L_z$ (see Eq. 18).

5. Size of disturbances and anelastic approximation

We discuss now three kinds of g-mode oscillations, which result from either large-scale or small-scale initial disturbances and from different ω_0 controlling the coupling between p- and g-modes. In Fig. 1 we have plotted $\omega_{p/g}(k_r)$ for $k_z = 0$ and different ω_0 (i.e. different γ). We compare at the same time with the

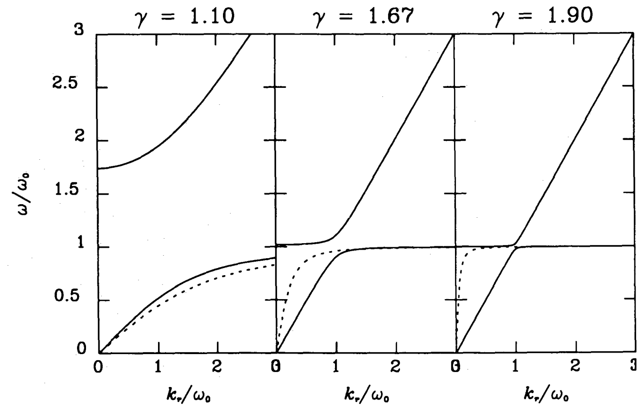

Fig. 1. The dispersion relation $\omega = \omega(k_r)$ for three different γ and $k_z = 0$. Both axes are normalized to ω_0 (= 0.7, 4.9 and 19 for the three cases). The solid curves are the p-mode (uppermost) and the g-mode branch from Eq. (35). The dashed line refers to the anelastic approximation (Eq. 45). Note the different behaviour of the curvature between ω_g and $\omega_{\rm anel}$ for $\omega < k_r$. The anelastic approximation is useful if γ is close to unity (close to marginal convective stability), but breaks down for larger values of γ and small k_r (large wavelength)

corresponding dispersion relation for the anelastic approximation $\omega_{\rm anel}(k_r)$ (Gough, 1969), which can be obtained by dropping the time derivative in Eqs. (3), (5), and (11). Instead of Eq. (35) we have:

$$\omega_{\rm anel}^2 = \omega_g^2 k_r^2/(k^2+1) \tag{44}$$

From Fig. 1 we may find the following properties of ω_g and $\omega_{\rm anel}$:

i) $\omega_g(k_r)$ increases linearly if $k_r < \omega_0$ and ω_0 is large (see right panel)

(ii) strong deviations from the linear increase of ω_g for $k_r < \omega_0$ but small ω_0 (left panel) and of $\omega_{\rm anel}(k_r)$ for $k_r < \omega_0$ and arbitrary ω_0

iii) $\omega_g \approx$ const for $k_r > \omega_0$ and arbitrary ω_0 (all three panels)

We thus expect also three different types of g-mode oscillations. For disturbances with a size larger than:

$$l_{\rm crit} \approx 2\pi\omega_0^{-1}H_0 = (2\pi\gamma/\sqrt{\gamma-1})H_p \tag{45}$$

(in physical units) there can be horizontal wave propagation either with or without dispersion (case i and ii respectively). Smaller disturbances lead to Brunt-Väisälä oscillations without, however, producing propagating waves (case iii). In Fig. 2a–4a we give examples of the velocity fields for the three cases ii, i, and iii. Comparison is made with the anelastic approximation (Fig. 2b–4b), for which we obtained the following solution:

$$\begin{pmatrix} v_r' \\ v_z' \\ s' \\ p' \end{pmatrix} = \frac{R_r^2 R_z}{2\sqrt{\pi}}\int_0^\infty dk_r \int_0^\infty dk_z \exp(-k_r^2 R_r^2/4 - k_z^2 R_z^2/4) \times$$

$$\begin{pmatrix} k_r^2 J_1(k_r r)(\cos k_z z + k_z \sin k_z z)(1+k^2)^{-1}(\omega_{\rm anel}/\omega_0)^{-1}\sin\omega_{\rm anel}t \\ k_r J_0(k_r r)\cos k_z z(\omega_{\rm anel}/\omega_0)\sin\omega_{\rm anel}t \\ k_r J_0(k_r r)\cos k_z z\cos\omega_{\rm anel}t \\ k_r J_0(k_r r)(\cos k_z z + k_z\sin k_z z)(1+k^2)^{-1}\omega_0\cos\omega_{\rm anel}t \end{pmatrix} \tag{46}$$

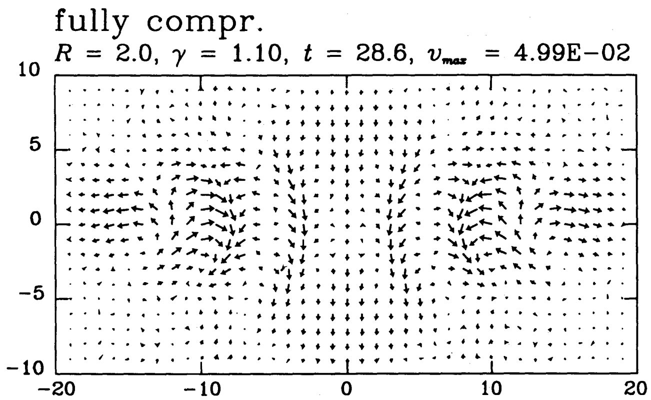

Fig. 2a. A snapshot of the velocity field at the time when the bubble has completed three Brunt-Väisälä oscillations, i.e. $\omega_0 t = 6.4\pi$ ($t = 28.6$). The radius of the bubble is $R_r = R_z = 2.0$ and $\gamma = 1.1$, which corresponds to the case (ii) in Sect. 5

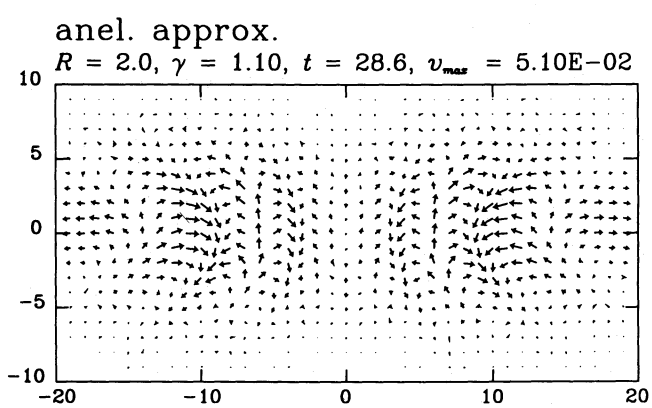

Fig. 2b. Same as Fig. 2a, but using the anelastic approximation. The differences between the anelastic and the fully compressible computation are only minor

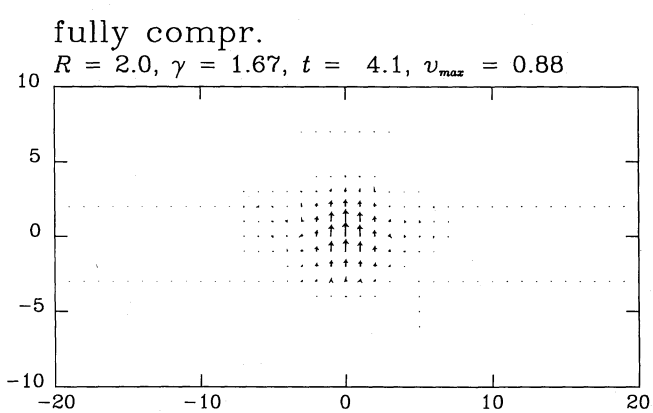

Fig. 3a. Same as Fig. 2a, but for $\gamma = 5/3$. The time corresponding to $\omega_0 t = 6.4\pi$ is much shorter in this case ($t = 4.1$)

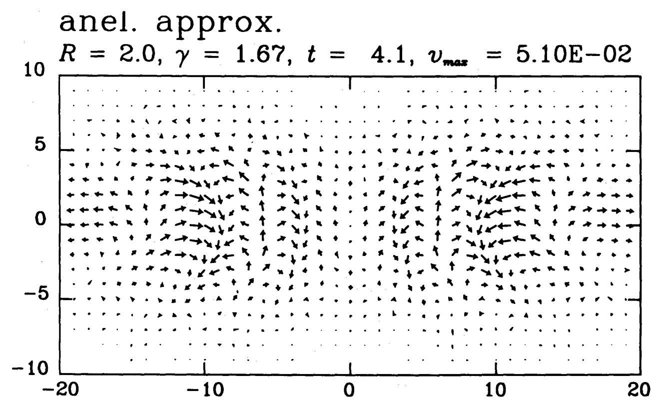

Fig. 3b. Same as Fig. 3a, but using the anelastic approximation. The flow pattern is similar to the case $\gamma = 1.1$ (see Fig. 2a, b). However, the discrepancy with respect to the fully compressible case is quite remarkable

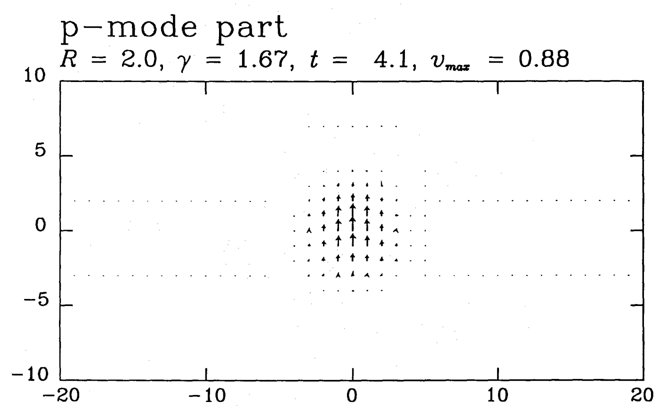

Fig. 3c. Same as Fig. 3a, but only the p-mode part is plotted. The flow is now dominated by acoustic oscillations at the Lamb cutoff frequency $\sqrt{(\omega_0^2 + 1)}$. The p-modes cannot propagate, since there is no group velocity for small wave numbers (large bubble), see also Fig. 1, right and middle panel

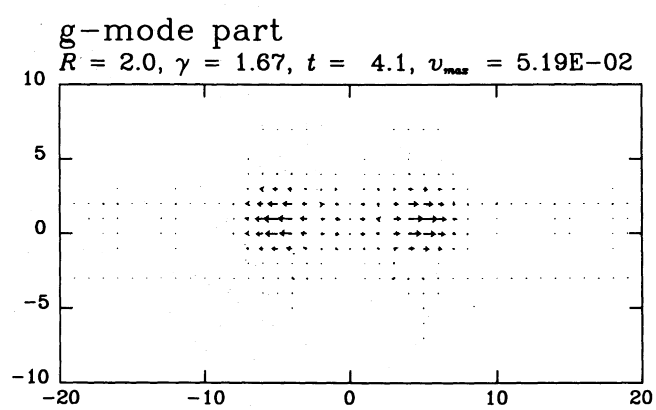

Fig. 3d. Same as Fig. 3a, but only the g-mode part is plotted. This picture has now nothing in common with the flows in the figures above. The initial disturbance propagates sideways without any dispersion. This is because the curvature in the dispersion relation (see Fig. 1, right and middle panel) is very small in contrast to the curvature for the anelastic approximation

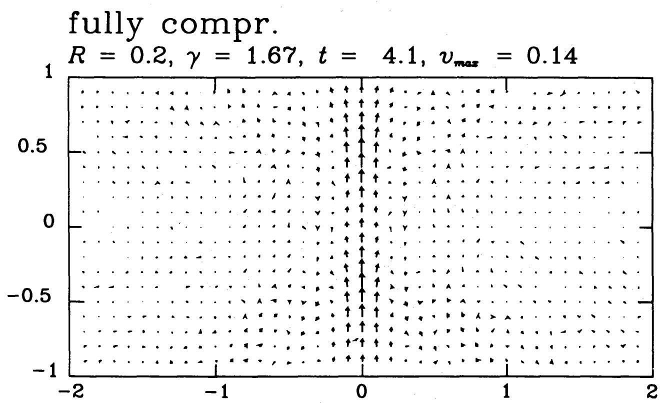

Fig. 4a. Same as Fig. 3a, but starting with a small bubble ($R_r = R_z = 0.2$)

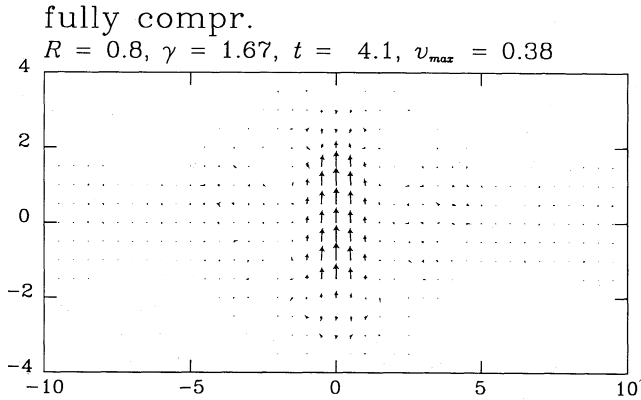

Fig. 5a. Same as Fig. 3a and 4a, but starting with a bubble of intermediate radius close to l_{crit} ($R_r = R_z = 0.8$). Note that the vertical extension of the velocity pattern is about half way between the case 3a and 4a

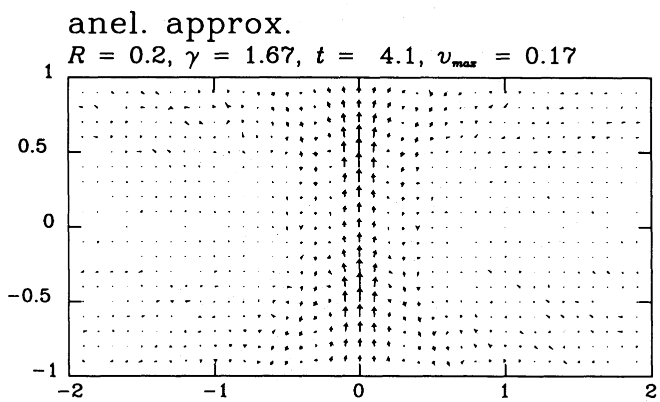

Fig. 4b. Same as Fig. 4a, but using the anelastic approximation. The differences between the anelastic and the fully compressible case are again only minor ones, as in the case $\gamma = 1.1$ (see Fig. 2a, b)

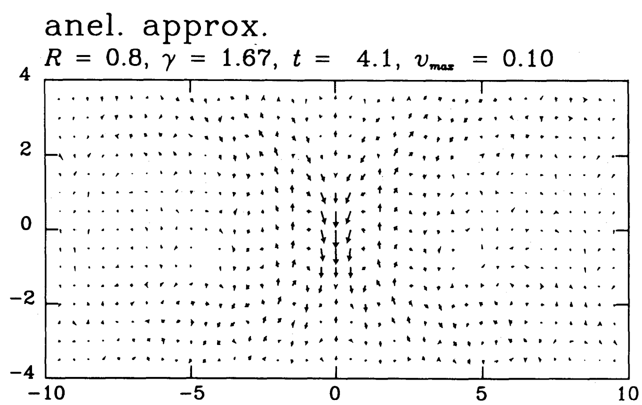

Fig. 5b. Same as Fig. 5a, but using the anelastic approximation

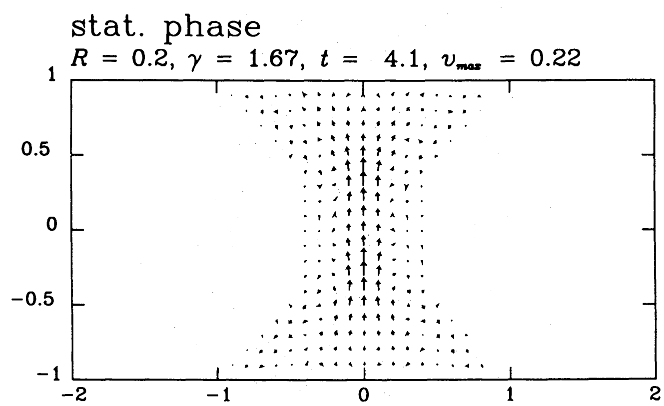

Fig. 4c. Same as Fig. 4b, but the solution is computed using the stationary phase (Sect. 6)

The integrals (42) and (46) have been calculated by quadrature. From Fig. 3a and 3b we see that the anelastic approximation breaks down, if γ is not any longer close to unity, i.e. if ω_0 large. The physical reason for this is that the group velocity of the gravity waves is close to the speed of sound. Fig. 5a, b give an example for an intermediate radius.

6. An approximate solution for small bubbles

Some properties of a bubble undergoing Brunt-Väisälä oscillations can be obtained from the theory of harmonic oscillations (acceleration × mass = restoring force). Quantities such as period and maximum velocity have been sometimes considered to test hydrodynamic computer codes (Stefanik et al., 1984; Gigas and Steffen, 1984; Schmitz, 1986). However, there are other basic features of the flow, for example the stirring up of the atmosphere, found by Stefanik et al. (1984). The intention of this section is, therefore, to provide some explicit results describing the vertical growth of the velocity field (see Fig. 4a) and the phase relations.

We consider the flow due to a disturbance by a small bubble ($R_r \ll 1$ and $R_z \ll 1$) after a time sufficiently long for the acoustic waves to propagate out of the region of interest. We may therefore neglect the pressure response $G^{(p)}$ in Eq. (37). Moreover, we shall make use of the anelastic approximation, which proved to be useful, if the size of the disturbance is small (see Sect. 5). We shall demonstrate the procedure for the entropy integral in Eq. (46), which is the simplest one and reduces with the anelastic approximation to:

$$s(t,r,z) = \mathrm{Re}\int_0^\infty \frac{dk_r}{2\pi} J_0(k_r r) k_r \int_{-\infty}^\infty \frac{dk_z}{2\pi} \exp(i\Phi) q_{04}(\boldsymbol{k}) \tag{47}$$

with a phase Φ:

$$\Phi = k_z z - \omega_0 t k_r (k_r^2 + k_z^2 + 1)^{-1/2} \tag{48}$$

Consider first the k_z-integral. If the time is large, the term $\exp(i\Phi)$ becomes very oscillatory with respect to k_z and k_r, thus leading partly to destructive interference. The function $q_{04}(\boldsymbol{k})$ is at the same time only slowly varying. The integral has its main contribution for $k_z = k_{z0}$ where the phase becomes stationary i.e. $\partial\Phi/\partial k_z = 0$.

The method of stationary phase, often used in optics and hydrodynamics (see e.g. Sommerfeld, 1945), may be applied to the k_z-integral in Eq. (46). It consists of a Taylor expansion of the phase up to terms of second order. The k_z-integral then becomes approximately:

$$k_z\text{-integr.} \approx \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \exp i[\Phi(k_{z0}) + \tfrac{1}{2}(k_z - k_{z0})^2 \Phi''] q_{04}(k_{z0})$$
$$= [-2i\Phi''(k_{z0})]^{-1/2} \exp i\Phi(k_{z0}) q_{04}(k_{z0}) \tag{49}$$

The k_z-derivatives of the phase Φ, denoted by a prime, are found to be:

$$\Phi' = z + \omega_0 t k_r k_z (k_r^2 + k_z^2 + 1)^{-3/2} \tag{50}$$

$$\Phi'' = -\omega_0 t k_r (k_r^2 - 2k_z^2 + 1)(k_r^2 + k_z^2 + 1)^{-5/2} \tag{51}$$

In Eqs. (48), (50) and (51) we now assume $k_r \gg 1$, because in the k_r-integral also destructive interference appears at small k_r (note the slope for small k_r in Fig. 1). We find the extremum to be then approximately at:

$$k_{z0} \approx -z k_r^2/\omega_0 t \tag{52}$$

and Eqs. (48) and (51) give at this point:

$$\Phi(k_{z0}) \approx -z^2 k_r^2/\omega_0 t - \omega_0 t \tag{53}$$

$$\Phi''(k_{z0}) \approx -k_r^{-2} \omega_0^t \tag{54}$$

For sufficiently small z-values, satisfying

$$z k_r/2 \approx z/R_r \ll \omega_0 t \tag{55}$$

the influence of k_{z0} in the source term may be neglected. Inserting now Eq. (49), (53) and (54) into (47), the resulting k_r-integral may be solved in terms of the confluent hypergeometric function M (see Abramowitz and Stegun, Eq. 11.4.28, 1970):

$$s'(t, r, z) = \mathrm{Re}\tfrac{1}{2}\left(\frac{2\pi i}{\omega_0 t}\right)^{1/2} \exp(-i\omega_0 t)\frac{R_z}{R_r} a^3 M(\tfrac{3}{2}, 1, -a^2 r^2/R_r^2) \tag{56}$$

with a complex function

$$a = a(t, z) = [1 + 4i(z/R_r)^2/(\omega_0 t)]^{-1/2} \tag{57}$$

The real part of Eq. (56) can be evaluated by using some elementary algebra. Properties of this solution are summarized below:

i) The bubble oscillates with the frequency ω_0, but a phase shift of $\pi/4$ appears (note: $i^{1/2} = \exp i\pi/4$).

ii) The amplitude decreases algebraically $\propto t^{-1/2}$ and is proportional to the ratio R_z/R_r.

iii) The phase of the flow along the axis above and below the plane $z = 0$ advances the flow in the equatorial plane, because of same sign of t and $a(t, z)$ in the exponent of s', which can be written in the form: $s' \propto \mathrm{Re}\exp -i[\omega_0 t + 3/2 \operatorname{arctg} \mathrm{Im}(a^{-2})]$.

iv) The vertical extension of the oscillating regions increases with time $\propto t^{-1/2}$, because the solution depends only on the combination z^2/t.

In the same manner we also obtain the other components of the vector as follows:

$$\begin{pmatrix} v_r' \\ v_z' \\ s' \\ p' \end{pmatrix} = \mathrm{Re}\tfrac{1}{2}\left(\frac{2\pi i}{\omega_0 t}\right)^{1/2} \exp(-i\omega_0 t)\frac{R_z}{R_r} a^3 \times$$
$$\begin{bmatrix} ir\left[\frac{1}{2}M\left(\frac{3}{2}, 2, \hat{r}^2\right) + 6iz/(\omega_0 t)\hat{a}^2 M\left(\frac{5}{2}, 2, \hat{r}^2\right)\right] \\ iM\left(\frac{3}{2}, 1, \hat{r}^2\right) \\ M\left(\frac{3}{2}, 1, \hat{r}^2\right) \\ \omega_0\left[\frac{1}{2}\hat{a}^{-2} M\left(\frac{1}{2}, 1, \hat{r}^2\right) + 2iz/(\omega_0 t) M\left(\frac{3}{2}, 1, \hat{r}^2\right)\right] \end{bmatrix} \tag{58}$$

We have used here the following abreviations:

$$\hat{r}^2 = -a^2 r^2/R_r^2 \qquad \hat{a}^2 = a^2/R_r^2 \tag{59}$$

An example of this solution is plotted in Fig. 4c and 6a, b. The hypergeometric function M with complex argument was calculated using a series expansion (Abramowitz and Stegun, Eq. 13.1.2, 1970).

7. Conclusions

A Green's function tensor is derived giving the response of an isothermal atmosphere to small disturbances representing deviations from a state of static stratification. Studying the effect of different sizes of the initial disturbance, we find that horizontally propagating gravity waves can be excited only by large scale disturbances (size of the supergranulation in the case of the Sun). The question whether such long waves could be important or

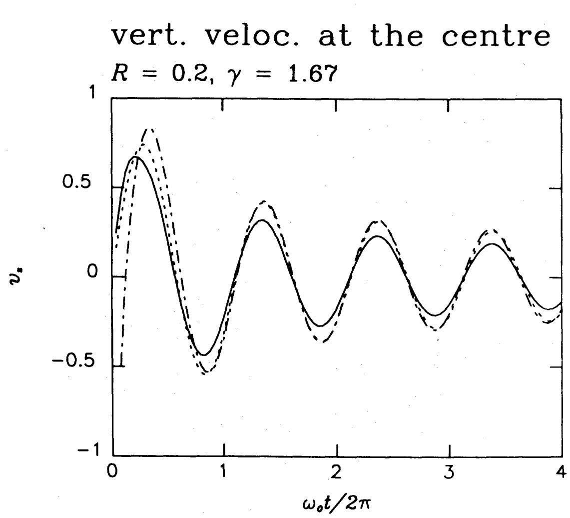

Fig. 6a. The vertical velocity at the centre ($r = z = 0$) versus time for the case of Fig. 4a (solid line). The dashed line refers to the anelastic approximation and the dashed-dotted one to the method of stationary phase which satisfies a $t^{-1/2}$ law (see Sect. 6, ii)

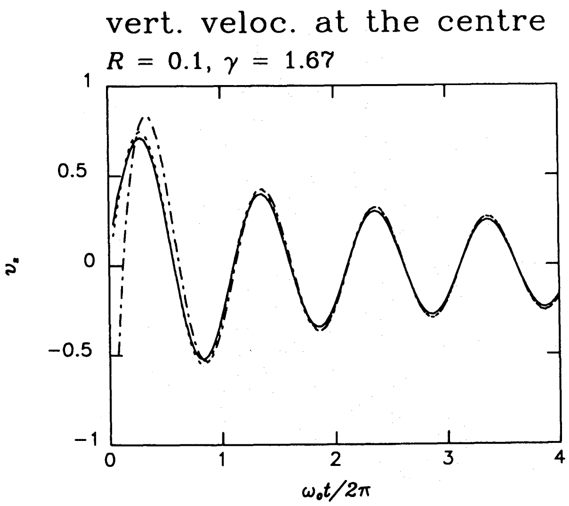

Fig. 6b. Same as Fig. 6a, but with a smaller radius $R_r = R_z = 0.1$ instead of 0.2. The anelastic approximation agrees in this case also quantitatively with the fully compressible computation

observable in the Sun remains open. Since the radiative cooling time is very short in the upper photosphere, they may perhaps be hard to maintain there. In the solar interior or the overshoot region below the bottom of the convection zone gravity waves can be excited, as numerical simulations by Hurlburt et al. (1986) suggest. However, effects of the spherical geometry become important, because the critical length (ten pressure scale heights) is already of the order of the radius of the Sun. In the case of small scale disturbances the anelastic approximation gives correct results and the method of stationary phase appears then to be a useful tool for understanding the properties of the flow. The damping rate and the variation of the velocity field, observed by Stefanik et al. (1984), can be confirmed quantitatively by this method.

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