The stability of nonlinear dynamos and the limited role of kinematic growth rates

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Summary. The physical significance of growth rates of kinematic dynamos is discussed in the context of the observation that usually a magnetic field of a single symmetry dominates in the Sun and other cosmic objects. It is concluded that these growth rates are not the decisive factor determining the final state of the field. The possibility that the stability of different solutions of nonlinear dynamos determines the final state is investigated with the help of several models. The examples of simple $\alpha^2$-dynamos investigated show that, in spite of the asymptotic equality of the kinematic growth rates, usually the only solution which remains stable is that with the smallest marginal dynamo number. Dynamo models in spherical geometry are found, however, in which both symmetric and antisymmetric solutions are stable. The kind of symmetry finally established depends in these cases on the initial conditions, i.e. on the history of the object. In no case was a steady solution found that was a superposition of the two distinct symmetry types, that is a non-symmetric steady final state was never reached. However, in connection with the investigations of the oscillatory dynamo we discovered a case where both the symmetric and the antisymmetric solutions are unstable. The attractor in this case is a torus: non-symmetric quasiperiodic solutions oscillate between the unstable symmetric and antisymmetric solutions with a long period.

Key words: hydromagnetic dynamos – nonlinear stability – the Sun: magnetic field

1. Introduction

The typical astrophysical dynamo is modelled by an electrically conducting rotating sphere, the internal structure and the motions of which show symmetry with respect to the rotational axis and to the equatorial plane. In the kinematic case a model of that kind excites eigenmodes of different symmetry types. The fields will either show symmetry or antisymmetry with respect to the equatorial plane. We denote them by $S$ and $A$, and will also speak about even ($S$) and odd ($A$) modes.

The essence of kinematic mean-field dynamos may be found in stability maps, which display the lines of zero growth rates for the different eigenmodes $B_n$. An example is given in Fig. 1. $C_s$ and $C_a$ are dimensionless parameters characterizing the $z$-effect and differential rotation. In the notation $Am$ and $Sm$, the number $m$ corresponds to the dependence on the longitude by $\cos\theta$.

Clearly, one wishes to know which of these eigenmodes will be realised for the cosmic object considered. Obviously this depends where the object is situated in that diagram. In region (i), where only one mode has a positive growth rate, a unique answer can be given: a field of type $A0$ will be excited. In region (ii) a field of type $S0$ has also a positive growth rate and will compete with the $A0$-field. Region (iii) represents a much more complicated, already highly nonlinear, situation where non-axisymmetric modes will also grow.

It is widely assumed that in the competition between different $B$-modes the one with the largest growth rate wins. In this way a criterion seems to exist and may be checked by observation. However, calculations of growth rates carried out earlier (Yoshimura et al., 1984) have revealed that a significant difference between the growth rates of fields with different parity exists only close to the critical dynamo number. All numerical examples studied so far show that the growth rates of the first odd parity mode and the first even one are asymptotically equal for large dynamo numbers. In this way practically no decision is possible unless one assumes that the dynamo number of the cosmic object, e.g. the Sun, is close to the critical value.

We investigate here the behaviour of the growth rates of further models. First we consider dynamo models in which this problem can be treated analytically. Then we calculate numerically solar-type $\alpha\omega$-dynamos. In all cases we will find the growth rates of the first modes of even and odd parity are asymptotically equal.

The observational results from the Sun and the planets Earth, Jupiter, and Saturn reveal that these objects excite magnetic fields with a clear dominance of one parity, in these cases the odd one. In the Sun this is most clearly manifested by Hale's polarity law revealing the dominance of the odd parity for the toroidal field. Stenflo and Vogel (1986) furthermore derived the result that only the odd parity of the poloidal field has a clear 22-year cycle.

Consequently we have to conclude that the criterion based on the growth rates does not apply. A quite different criterion was formulated by Krause and Meinel (1988), who postulated that the stability of nonlinear solutions determines which field is finally excited by an object. We follow here this approach and

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We see that, indeed, for all modes the growth rates have an asymptote which is independent of the order of the multipole $n$. The asymptote depends on the number $l$ of knots in the radial direction only.

3. A one-dimensional $a^2$-model

We will consider now the basic equations

\[ \frac{\partial B_x}{\partial t} = \text{curl}(zB_y) + (\mu \sigma)^{-1} \Delta B \]

inside a plane layer ($-d \leq z \leq d$);\n
\[ \text{curl} B = 0 \] outside the layer ($|z| > d$); and\n
\[ \text{div} B = 0 \]
everywhere. $B$ is continuous at the boundaries of the layer and vanishes as $|z| \to \infty$. We seek solutions of the form $B = B(z, t)$. This leads to the equations

\[ \frac{\partial B_x}{\partial t} = -\frac{\partial (zB_y)}{\partial z} + (\mu \sigma)^{-1} \frac{\partial^2 B_x}{\partial z^2} \]

\[ \frac{\partial B_y}{\partial t} = \frac{\partial (zB_x)}{\partial z} + (\mu \sigma)^{-1} \frac{\partial^2 B_y}{\partial z^2} \]

for the nonvanishing components $B_x(z, t)$ and $B_y(z, t)$, with the boundary conditions

\[ B_x(d, t) = B_x(-d, t) = B_y(-d, t) = 0. \]

We note that Eqs. (10) and (11), with $B_x$ and $B_y$ replaced by $B$ and $B_y$, are often used as a simple model to study axisymmetric disk dynamos within the “local approximation” (cf. Zeldovich et al., 1983). However, this approximation is not well justified, as mentioned by Radler and Brauer (1987). In particular the use of the boundary condition (11) is not correct in that context. We therefore consider the model (10), (11) only as a mathematical idealization (one-dimensional reduction) of the 3-dimensional $a^2$-dynamo problem.

Introducing dimensionless coordinates $\zeta$, $\tau$, defined by

\[ t = \mu \sigma d^2 \tau, \quad z = z\zeta, \]

and the complex function

\[ B(\zeta, \tau) = B_x + iB_y, \]

we obtain the equation

\[ \dot{B} = B^* + iC_\zeta(\bar{a} f B'), \quad C_\zeta = \mu \sigma a_0 d, \]

with the boundary conditions

\[ B(1, \tau) = B(-1, \tau) = 0. \]

Here we have assumed

\[ a = a_0 \tilde{a}(z) f(BB^*) \]

where $a_0$ is a positive constant, $\tilde{a}(z)$ a prescribed function of $z$ and $f(BB^*)$ a prescribed function modelling the back-reaction of the magnetic field on the turbulence.

Now we consider the special case

\[ \tilde{a} = \begin{cases} 1 & \text{for } 0 < \zeta < 1 \\ 0 & \text{for } -1 < \zeta < 0 \end{cases} \]

and discuss first the kinematic problem, i.e. $f = 1$. The general solution of the problem (14), (15), (17) is given by

\[ B = B^{A} + B^{S}, \]

\[ A = \frac{1}{2} \left[ 1 + \text{erf}\left(\frac{z}{2\sqrt{\tau}}\right) \right] \quad \text{and} \quad S = \frac{1}{2} \left[ 1 - \text{erf}\left(\frac{z}{2\sqrt{\tau}}\right) \right]. \]
where

\[ B^{(A)}(-\zeta, \tau) = -B^{(A)}(\zeta, \tau), \quad B^{(S)}(-\zeta, \tau) = B^{(S)}(\zeta, \tau). \]  

(19)

\[ B^{(A)}(0, \tau) = 0, \quad B^{(A)}(1, \tau) = 0, \]  

(20)

\[ B^{(S)\prime}(0, \tau) + iC_\omega B^{(S)}(0, \tau) = 0, \quad B^{(S)}(1, \tau) = 0. \]  

(21)

The conditions at \( \zeta = 0 \) follow from the continuity of \( B \) and \( B + iC_\omega \partial B \) (see Eq. (14)) together with Eq. (19). These solutions can be found in the form

\[ B^{(A)} = \sum_{n=1}^{\infty} c_n^{(A)} e^{i\lambda_n^{(A)} \tau} B_n^{(A)}(\zeta), \]  

(22)

\[ B^{(S)} = \sum_{n=1}^{\infty} c_n^{(S)} e^{i\lambda_n^{(S)} \tau} B_n^{(S)}(\zeta), \]  

(23)

where \( c_n^{(A)} \) and \( c_n^{(S)} \) are arbitrary complex constants determined by the initial conditions.

The antisymmetric modes (for \( 0 \leq \zeta \leq 1 \)) are given by

\[ B_n^{(A)}(\zeta) = e^{-iC_\omega \zeta/2} n n \zeta, \]  

(24)

with the eigenvalues

\[ \lambda_n^{(A)} = C_\omega^2 / 4 - n^2 \pi^2. \]  

(25)

The symmetric modes are (\( 0 \leq \zeta \leq 1 \))

\[ B_n^{(S)}(\zeta) = e^{-iC_\omega \zeta/2} \sin(\delta_n(1 - \zeta)) \]  

(26)

with the eigenvalues

\[ \lambda_n^{(S)} = C_\omega^2 / 4 - \delta_n^2, \]  

(27)

where the \( \delta_n \)s have to be determined as nontrivial solutions of the transcendental equation

\[ \delta_n = \frac{i}{2} C_\omega \tan \delta_n. \]  

(28)

Without loss of generality we assume \( \text{Re} \delta_n > 0 \), and arrange the solutions according to

\[ (2n - 1) \pi/2 < \text{Re} \delta_n < n \pi. \]

In contrast to the antisymmetric modes, the symmetric modes have always complex eigenvalues \( \lambda_n^{(S)} \). The growth rates \( \text{Re} \lambda_n \) for the first modes are depicted in Fig. 2. The first S-mode is growing for \( C_\omega > 4.0066 \), the first A-mode for \( C_\omega > 2\pi \). For large \( C_\omega \) the growth rates become indistinguishable. This can also be seen by comparing the asymptotic expansion of (27), (28)

\[ \lambda_n^{(S)} = C_\omega^2 / 4 - n^2 \pi^2 + 4n^2 \pi^2 / C_\omega + 12n^2 \pi^2 / C_\omega^2 + O((C_\omega^{-3})) \]  

(29)

with (25), i.e.

\[ \text{Re} \lambda_n^{(S)} - \text{Re} \lambda_n^{(A)} = 12n^2 \pi^2 / C_\omega^2 + O(C_\omega^{-3}). \]  

(30)

Thus the results of the kinematic analysis can be summarized as follows: For \( C_\omega < 4.0066 \) no dynamo activity is possible, any initial magnetic field will decay. For \( 4.0066 < C_\omega < 2\pi \) a symmetric magnetic field mode will grow. For \( C_\omega > 2\pi \) symmetric and antisymmetric modes are growing and a prediction of the final state is impossible on the basis of the kinematic results alone.

### 4. Growth rates for zω-dynamo of solar type

In this section we present results of numerical calculations concerning the growth rates of some zω-dynamo. We will find further evidence that the asymptotic equality will also hold here, although a rigorous proof is not available.

First we investigate the growth rate of the first odd parity mode and the first even mode for the Steenbeck–Krause model 1 (Steenbeck and Krause, 1969). The results are given in Table 1. \( \lambda \) denotes the growth rate and \( \Omega_\alpha \) the frequency of the magnetic cycle. Both quantities are here dependent only on the dynamo number \( C_\alpha C_\omega \), where \( C_\alpha \) is determined as before and \( C_\omega \) is a dimensionless measure of the differential rotation. We find that the magnetic field \( B \equiv 0 \) becomes unstable for \( C_\alpha C_\omega \approx 20.7 \times 10^3 \) and here a field of type \( A0 \) with a dipole parallel to the axis of rotation.

| Table 1. The frequency and growth rate for model 1 of Steenbeck and Krause (1969) for different values of the dynamo number \( C_\alpha C_\omega \). These eigenvalues in this and the other tables were computed with 100 radial gridpoints and 16 spherical harmonics. |
|---|---|---|---|
| \( 10^{-3} C_\alpha C_\omega \) | \( \pm \Omega_\alpha \) | \( \lambda \) | \( \pm \Omega_\alpha \) | \( \lambda \) |
| 0.0 | 0.0 | -9.9 | 0.0 | -20.2 |
| 10.0 | 22.2 | -8.0 | 20.4 | -17.1 |
| 20.0 | 31.2 | -0.4 | 29.6 | -6.3 |
| 30.0 | 37.9 | 5.9 | 36.5 | 1.3 |
| 40.0 | 43.2 | 11.3 | 42.1 | 7.5 |
| 50.0 | 47.8 | 16.2 | 46.9 | 12.8 |
| 60.0 | 51.9 | 20.6 | 51.2 | 17.5 |
| 80.0 | 59.1 | 28.3 | 58.6 | 25.7 |
| 100.0 | 65.4 | 35.1 | 65.2 | 32.8 |

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rotation will start to grow. From $C_x C_\alpha \approx 28.3 \times 10^3$ on, a field of type S0 will also grow, but the growth rate of the A0-type field remains larger.

When the $\alpha$-effect producing the toroidal field from the poloidal one is not neglected compared with effects of differential rotation, the asymptotic equality of A0 and S0 modes is more clearly demonstrated. This can be seen from Table 2 and Fig. 3, where we present the results for this case using model I of Steenbeck and Krause with $C_\alpha = +1000$. The growth rates of the two fields become indistinguishable for $C_x > 40$. The results for a further model with $\alpha$ and $\omega$ profiles as in Rädler (1986, Fig. 16) are presented in Table 3a and Fig. 4. The behaviour of the growth rates is here more complicated. For example the growth rates of the S-type solution exceed that of the antisymmetric one for certain $C_x$ values. For even higher $C_x$ the S0-mode splits into two non-oscillatory solutions. This does not mean that a new eigenvalue emerged because all axisymmetric oscillatory modes have a pair of complex conjugate eigenvalues. Also the A0-solution becomes non-oscillatory, but the second eigenvalue transforms with another non-oscillatory one to a pair of complex conjugate oscillatory modes.

Table 3a. The frequency and growth rate for different $C_x$ of a model with $\alpha$ and $\omega$ profiles as in Rädler (1986, Fig. 16) with $C_\alpha = -1000$. See also Fig. 4. For $C_x > 3.2$ the first modes are only non-oscillatory and their growth rates become asymptotically equal.

<table>
<thead>
<tr>
<th>$C_x$</th>
<th>$\pm \Omega_\lambda$</th>
<th>$\lambda$</th>
<th>$\pm \Omega_\lambda$</th>
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<td>0.0 -9.9</td>
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<tr>
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</tr>
<tr>
<td>0.5</td>
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<td>.0 -8.2</td>
<td>11.9 -11.5</td>
<td>.0 -8.2</td>
</tr>
<tr>
<td>1.0</td>
<td>15.7 -7.5</td>
<td>.0 -5.2</td>
<td>15.7 -7.5</td>
<td>.0 -5.2</td>
</tr>
<tr>
<td>1.5</td>
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<td>.0 -3.7</td>
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<td>.0 -3.7</td>
</tr>
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<td>2.5 -6.7</td>
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<td>2.5 -6.7</td>
</tr>
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<td>18.1 4.9</td>
<td>9.6 -3.0</td>
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<tr>
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<td>9.9 -7.2</td>
<td>16.7 7.1</td>
<td>9.9 -7.2</td>
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<tr>
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<td>0.0 46.0</td>
<td>0.0 45.9 14.6 7.2</td>
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<tr>
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<tr>
<td>4.00</td>
<td>0.0 94.5 0.4 40.2</td>
<td>0.0 94.5</td>
<td>0.0 94.5 0.4 40.2</td>
<td>0.0 94.5</td>
</tr>
</tbody>
</table>

Table 3b. The frequency and growth rate of the first non-axi-
symmetric modes A1 and S1 for some values of $C_x$ using the same model as in Table 3a. Note that the growth rates of both symmetries are always very close to each other.

<table>
<thead>
<tr>
<th>$C_x$</th>
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<th>$\lambda$</th>
<th>$\Omega_\lambda$</th>
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<td>999.3</td>
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<td>3.2</td>
<td>999.4 19.8</td>
<td></td>
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</tr>
</tbody>
</table>

Fig. 3. Growth rates of the first A- and S-type solutions (solid and broken lines, respectively) for the same model as in Fig. 1 for $C_\alpha = +1000$. The $\alpha$-effect producing the toroidal field from the poloidal one is not neglected here, in contrast to the model used in Table 1.

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the fields of type $S1$ and $A1$, although with an asymptote which differs from that of $A0$ and $S0$ (see Table 3b). These numerical results confirm the conjecture that no decision between odd and even parity fields can be based on the considerations of the growth rates of the linear eigenmodes.

5. Nonlinear analysis of the one-dimensional $x^2$-model

The dynamo model treated in Sect. 3 allows the reduction of the solution of the nonlinear steady problem to a quadrature. It can easily be shown that antisymmetric steady nonlinear solutions are possible. The boundary conditions (20) reduce the problem for $0 \leq \xi \leq 1$ to the model discussed by Krause and Meinel (1988). While symmetric steady solutions are impossible, solutions of the form

$$B^{(S)}(\xi, \tau) = \tilde{B}(\xi) e^{i \Omega_* \tau}$$  \hspace{1cm} (31)

can be found, where $\Omega_*$ is a real constant to be determined as the eigenvalue of the ordinary differential equation

$$i \Omega_* B = \tilde{B}' + i \xi_s [f(B \tilde{B}^*) \tilde{B}]'$$  \hspace{1cm} (32)

following from (14) for $0 \leq \xi \leq 1$. The corresponding boundary conditions are

$$\tilde{B}' + i \xi_s f(B \tilde{B}^*) \tilde{B} |_{\xi=0} = 0, \quad \tilde{B}(1) = 0.$$  \hspace{1cm} (33)

The steady nonlinear solutions of $A$-type bifurcate from the trivial solution $B \equiv 0$ at the critical values $C^{(A)} = 2 \pi n$ and coincide there with the steady kinematic modes. Correspondingly the symmetric solutions (31) bifurcate at the marginal values $C^{(S)}$ of the symmetric oscillatory kinematic modes. At $C^{(S)}$ the eigenvalues $\Omega_{mn}$ of (32), (33) coincide with $\text{Im} \lambda^{(S)}_n$.

We considered more closely the case

$$f(0) = 1/(1 + BB^*).$$  \hspace{1cm} (34)

Numerical stability tests of the different nonlinear solutions have been carried out using (14) for $-1 \leq \xi \leq 1$ without symmetry restrictions. The first $S$-type solutions (bifurcating at $C^{(S)}$ = 4.0066, see Fig. 5a) proves to be stable against arbitrary perturbations. The first $A$-type solution (bifurcating at $C^{(A)} = 2 \pi$) is stable against antisymmetric perturbations but unstable against symmetric perturbations. All higher solutions of both types are unstable.

A particular stability test was based on the quantity $P$ defined by

$$P = \frac{[E^{(S)} - E^{(A)}]}{[E^{(S)} + E^{(A)}]},$$  \hspace{1cm} (35)

Fig. 5a. Bifurcation diagram for the nonlinear one-dimensional $x^2$-model. The $S$-type solution is oscillatory and the frequencies deviate quite substantially from those of the linear theory, if $C_\xi$ is large. The energy for the oscillatory solution is time independent

Fig. 5b. Evolution of $P$ for the one-dimensional $x^2$-model of Fig. 5a. For all different initial $\xi$ (or $P$) the solution turns into a pure symmetric one, i.e. $P = 1$. These curves were obtained by taking $C_\xi = 8$
where $E^{(S)}$ and $E^{(A)}$ are the energies of the symmetric and antisymmetric part of the $B$-field inside the conducting region. Note that $P = 1$ for a pure symmetric field and $P = -1$ for a pure antisymmetric one. The energy is defined here as

$$E^{(S/A)} = 1/2 \int_{\text{volume}} d^3x |B^{(S/A)}|^2$$

(36)

where $B^{(S)}$ and $B^{(A)}$ denote the symmetric and antisymmetric part of $B$.

We now test the stability of the $A$-type solution by adding a fraction $\varepsilon$ of the $S$-type solution to the $B$-field:

$$B_0 = B^{(A)} + \varepsilon B^{(S)}.$$  

(37)

Similarly the stability of the $S$-type solution is tested by taking

$$B_0 = B^{(S)} + \varepsilon B^{(A)}$$

(38)

as the initial condition.

Figure 5b gives the evolution of $P$ in the one-dimensional model for different initial disturbances obtained by step by step integration of $B$ in time for initial values of $P$ in the range $(-1, +1)$. Note that we used (37) for $P < 0$ and (38) for $P > 0$, and moreover that the initial condition is not fixed by $P$ alone but also by relative phase of $B^{(A)}$ and $B^{(S)}$. However, the qualitative nature of the results is not affected by this freedom of initial conditions. We find that the final stable solution has the value $P = 1$, i.e., the parity is symmetric. Our investigations show that the final state resulting from any initial field (except from the unrealistic situation of a pure antisymmetric initial field without any symmetric part) will always be given by the first $S$-type nonlinear solution. This solution can be considered as the nonlinear extension of the first marginal $S$-mode. We stress that this result of the clear prevalence of the $S$-type solution has nothing to do with the different kinematic growth rates.

6. Nonlinear dynamo models in spherical geometry

Analytical solutions for dynamo models in spherical geometry with latitudinally dependent $x$ are not available. We shall therefore study the stability of such models numerically, restricting ourselves, however, to the axisymmetric case. We solve the dynamo equations (7)–(9) on a two-dimensional grid employing a DuFort–Frankel time advance (see Proctor, 1977). When the calculation is performed in two quadrants of a meridional plane of a sphere, the boundary conditions do not select the parity, and fields of both parities can exist simultaneously.

6.1. Dependence of $x$ on the total energy

We now make a special choice of nonlinearity, where $x$ depends on the total magnetic energy $E = E^{(S)} + E^{(A)}$ (the "cross term" vanishes under the integral):

$$x = x(\theta) = C_x \cos \theta / (1 + E).$$

(39)

$C_x$ is the dynamo number and $x$ is independent of $r$ and changes sign at the equator. The bifurcation of the $S$- and $A$-type solutions from the trivial one appears at the critical dynamo numbers $C_x^{(S)} = 7.81$ and $C_x^{(A)} = 7.64$. These are in accordance with those calculated for the same model by Roberts (1972). For $C_x > C_x^{(S/A)}$, $|B|$ and so $E$ grow until the quantity $C_x / (1 + E)$ is reduced to the critical value, $C_x^{(S/A)}$, of the currently dominant mode. Then the steady state is reached. The energies are therefore determined by:

$$E^{(S/A)} = C_x / C_x^{(S/A)} - 1.$$  

(40)

The stability behaviour can readily be observed by following the time evolution of $P$, defined in Eq. (35). In Fig. 6a we have plotted the energy versus dynamo number for solutions of each parity. In Fig. 6b we have plotted $P$ versus time for different $\varepsilon$ such that the initial $P$ again covers the range from $-1$ to $+1$. Clearly the $A$-type solution is stable to symmetric disturbances and the $S$-type solution unstable to antisymmetric disturbances. This is further illustrated by an experiment in which the evolution of a purely $S$-type initial field was followed over many diffusion times. The field evolved to the steady $S$-type solution ($P = +1$) in time $t = O(1)$ with $C_x / (1 + E) = C_x^{(S)}$, and this configuration persisted until $t \gg 1$. Finally however this $S$-type solution changed.

![Fig. 6a. Bifurcation diagram for a spherical dynamo model with $x$ depending on the total energy $E$ only (see Eq. (39)). The energy increases linearly with $C_x$.](image1)

![Fig. 6b. Evolution of $P$ for the spherical dynamo model of Fig. 6a. For all different initial $\varepsilon$ (or $P$) the solution turns into a pure antisymmetric one, i.e. $P = -1$. These result was obtained using $C_x = 10$.](image2)
into the first $A$-type solution ($P = -1$), as the $A$-type noise inevitably present in the computed solution grew and eventually reduced $C_\alpha/(1 + E)$ to $C_\alpha^{(A)} < C_\alpha^{(S)}$. This is in agreement with the general result of Krause and Meinel (1988, Sect. 5). Note again that the mechanism is not just that of the solution with the fastest linear theory growth rate becoming exponentially larger than any other. The simple nonlinearity introduced quenches the symmetric solution by reducing an initially supercritical value of $C_\alpha$ to a value at which the symmetric solution decays and the antisymmetric solution is stable.

The typical time for the parity of the field to change can be estimated by means of the kinematic growth rates. A small initial antisymmetric contribution in a nearly symmetric field will increase with the growth rate $\lambda^{(A)}$. The effective $\alpha$ corresponds to the marginal value of the currently dominating parity, which is here the symmetric one with $C_\alpha^{(S)} = 7.81$. From linear theory we obtained for this value $C_\alpha$ the growth rate of the antisymmetric parity

$$\lambda^{(A)} = +0.55$$

when $C_\alpha = C_\alpha^{(S)}$.

For a given ratio of the initial energy of the antisymmetric disturbance $E_0^{(A)}$ and the final one $E^{(A)}$ we expect the typical timescale to be

$$\tau = \frac{1}{\lambda^{(A)}} \ln \left( \frac{E^{(A)}}{E_0^{(A)}} \right),$$

From Eqs. (35) and (40) we obtain

$$E^{(A)} = \frac{1 + P}{1 - P} \frac{C_\alpha^{(A)}}{C_\alpha^{(S)}} - 1$$

$$E_0^{(A)} = \frac{1}{1 - P} \frac{C_\alpha^{(S)}}{C_\alpha^{(S)} - 1}.$$

Considering now the uppermost curve in Fig. 6b, which starts from $P = 0.9$, we find from Eqs. (41)–(43) $\tau \approx 2.6$, which is indeed in accordance with Fig. 6b and with the series of snapshots in Fig. 7.

6.2. Dependence of $\alpha$ on the local energy density

The back-reaction of the magnetic field on the turbulence will in general reduce the $\alpha$-effect depending on the local strength of the field, rather than on the global energy, as assumed in the previous section. In order to keep $\alpha$ positive, we adopt here the frequently used expression

$$\alpha = \alpha(r, \theta) = C_\alpha \cos \theta/[1 + B(r, \theta)^2].$$

We have computed solutions for different dynamo numbers (see the bifurcation diagram in Fig. 8a) and have tested their stability in the same manner as described in the previous section. The evolution of the parity after disturbances with different $\varepsilon$ is plotted in Fig. 8b for $C_\alpha = 10$. We see that, in contrast to the previous case, there are now two stable solutions, one symmetric ($P = 1$) and the other antisymmetric ($P = -1$). Which of the possible solutions is realized depends on the initial condition. Similar results have been already obtained by Rädler (1984). From Fig. 8b it can be estimated that the “watershed” lies at
$P \approx +0.05$. This result is surprising, since the marginal dynamo numbers of both symmetries are well separated.

At first this result seems to be in contradiction to the general statement of Krause and Meinel (1988) that the second nonlinear solution, bifurcating from $C_1^{(S)}$ is in any case unstable. In their denotation $C_*^{(A)}$ has to be identified with $C_1$ and $C_*^{(S)}$ with $C_2$. However, according to their statement the unstable behaviour necessarily appears in a certain neighborhood of $C_*^{(S)}$ only, i.e. there may exist a certain value $C_* > C_*^{(S)}$ with the property that the symmetric solution is unstable for $C_* < C_*^*$, but stable for $C_* > C_*^*$.

In order to confirm this statement we calculated the stability for values of $C_2$ closer to $C_*^{(S)}$. It is found that the “watershed” tends, indeed, to $P=1$ if $C_2$ tends to $C_*^{(S)}$. For $C_2 = 7.89$ the “watershed” is very close to $P=1$ (Fig. 8c) and for $C_2 = 7.88$ no stable symmetric solution was found (Fig. 8d). Hence, the results can be expressed by an inequality:

$$7.81 = C_*^{(S)} < 7.88 < C_*^* < 7.89.$$  \hspace{1cm} (45)

The situation is illustrated in Fig. 8c.

We will now discuss how far we can understand the stability behaviour. If $B = B^{(S,A)} + b$ is substituted in the governing equation we find by linearising with respect to the perturbation $b$ the equation:

$$\frac{\partial b}{\partial t} = \text{curl} \left\{ \alpha [\sigma (r, 0) B^{(S,A)^2} b] + (\mu \sigma)^{-1} \Delta b + \text{curl} \left\{ \frac{\partial \alpha}{\partial B^2} B^{(S,A)} [B^{(S,A)} b] B^{(S,A)} \right\} \right\}.$$  \hspace{1cm} (46)
Where it not for the last term, the stability could be discussed in
terms of the growth rates derived from the kinematic dynamo
equation with modified $\alpha$. Then stability of the symmetric solu-
tion $B^{(a)}$ would be plausible if the maximal growth rate of
antisymmetric modes were still negative:

$$\lambda^{(a)} < 0, \quad \text{when} \quad \alpha = \alpha^{(b)} = \alpha(r, 0, B^{(s)})). \quad (47)$$

In the example considered above this was, however, not the case. For
instance, we found for $C_s = 10$:

$$\lambda^{(a)} = 0.62, \quad \text{when} \quad \alpha = \alpha^{(b)} \quad (48)$$

which would be large enough for the antisymmetric solution to
become important within the timespan plotted in Fig. 8b. Obviously,
the influence of the last term of the righthand side of
Eq. (46) provides an additional mechanism to stabilize the sym-
metric parallel solution. The sequence of snapshots in Fig. 7 (where
$\alpha$ is given by Eq. (39)) gives some insight into the process of
parity reversal. There is a stage with very little magnetic induction
in one hemisphere. Now, if the $\alpha$-feedback is $\theta$-dependent,
the local $\alpha$ in this hemisphere will be close to the maximal
possible value $\alpha_0$. This leads again to a field amplification in
this region. The direction of this field is still the same as before. The $\theta$-
dependent feedback is thus acting against a reversal of parity and
is stabilizing the symmetric solution.

The stability of the antisymmetric solution could simply be
explained by a condition similar to Eq. (47)

$$\lambda^{(b)} < 0, \quad \text{when} \quad \alpha = \alpha^{(a)} \quad (49)$$

In the example considered above ($C_s = 10$) we indeed found:

$$\lambda^{(b)} = -0.70, \quad \text{when} \quad \alpha = \alpha^{(a)} \quad (50)$$

That means, the last term in Eq. (46) does not qualitatively
change the stability behaviour of the antisymmetric solution.

6.3. Feedback on the mean velocity field

In the former sections we considered the back-reaction of
the magnetic field on the turbulent motions, which reduces the $\alpha$-
effect. There may be, however, also a considerable feedback on
the mean velocity field $u$, which can limit the magnetic energy.
Such dynamos have been investigated by Proctor (1977), who
computed strictly antisymmetric dynamos. We shall study here
the stability of both symmetric and antisymmetric solutions. We
restrict ourselves again to the axisymmetric case and assume
density $\rho$, kinematic viscosity $\nu$ and magnetic diffusivity $\eta$ to be
constant in a sphere of radius $R$.

The governing equations are now the induction and
momentum equations, which we solve simultaneously including
the mean motion in the induction equation:

$$\frac{\partial B}{\partial t} = \text{curl}[u \times B + \alpha B - \eta \text{curl} B]$$
$$\text{div} B = 0. \quad (51)$$

The momentum and continuity equations are:

$$\rho \frac{Du}{Dt} = -\nu p - 2\rho \Omega \times u + \rho g_{\text{eff}} - B \times \text{curl} B/\mu + \rho \nu \nabla^2 u$$
$$\text{div} u = 0, \quad (52)$$

where $u$ is the motion induced by the Lorentz force, $p$ is pressure,
$\Omega$ the (constant) angular velocity, and $g_{\text{eff}}$ the effective gravity.
We eliminate pressure and gravity by taking the curl of the
meridional part of the momentum equation. The obtained equations
were solved using step by step integration in time on a two-
dimensional grid with the same finite-difference representation
for the nonlinear terms as Proctor. We keep the Ekman number
$\nu/HR^2$ and the magnetic Prandtl number $\nu/\eta$ equal to unity and
take $\alpha = C_s \cos \theta$, with $C_s$ constant. For the velocity we employed
a stress-free boundary condition at the surface of the sphere.

In Fig. 9a we have plotted a bifurcation diagram for the
magnetic energies of both parities. The critical values for $C_s$ are,
of course, the same as in Sect. 6.1. We compared the energy of the
A-type solution with the values given by Proctor and found good
agreement.

We again studied the stability of these solutions by following
the evolution of $P$ for different disturbances. The results, which

![Fig. 9a. Bifurcation diagram for the spherical magneto-hydrodynamic
dynamo model of Sect. 6.3, where the equations for the mean velocity are
solved simultaneously with the induction equations. Note that the energy
of the S-type solution exceeds that of the A-type solution for $C_s > 8.2$.
Such a crossing of energies was not found for the dynamo models in
which the nonlinearity came only via $\alpha$](image)

![Fig. 9b. Evolution of $P$ for the spherical magneto-hydrodynamic
dynamo model of Fig. 9a. Similarly to the former case (Fig. 8b), the initial
condition determines the final symmetry. $C_s = 10.0$](image)
are presented in Fig. 9b, are similar to the case where the $x$-effect depends on the local energy density (Sect. 6.2), i.e., symmetric and antisymmetric solutions are each possible stable solutions. According to Lenz's rule we would also expect similar results to those obtained previously, since the main effect of this feedback is a local braking of the initially rigid rotation, creating thereby differential rotation, which gives rise to magnetic induction acting against the original field. This is inevitably a more or less dramatic simplification of a complicated physical system, whose complexity is likely to increase towards the highly nonlinear low viscosity regime.

6.4. Nonlinear oscillatory dynamos

In the last two sections we have considered two different nonlinear dynamo problems and found in each case conditions under which two solutions exist. This result is in contrast to those in Sects. 5 and 6.1, where only one solution was found to be stable. We argued then that the $\theta$-dependent feedback of the $x$-effect may stabilize the $50$-mode. This possible explanation could break down, however, if the magnetic field is oscillating, since the feedback (together with the local energy density) can become rather small during the cycle. This is also the case in the one-dimensional $\sigma$-model of Sect. 5, for which only one stable solution was found. In this section we follow this point in more detail.

Oscillatory solutions of the dynamo equations in spherical geometry are known, if differential rotation is present ($z\omega$-dynamo). We have studied the simple nonlinear model defined by Eq. (4) with an angular velocity $\Omega$ varying linearly with radius:

$$\Omega(r) = C_\omega/(\mu_0 R^2) \cdot r/R.$$  (53)

Keeping $C_\omega = -10^4 = \text{const}$, we found Hopf bifurcations from the trivial solution at the values $C_\omega^{(0)} = 0.549$ and $C_\omega^{(0)} = 0.728$ with the frequencies $\Omega_\omega^{(0)} = \pm 54.1$ and $\Omega_\omega^{(0)} = \pm 67.5$. These eigenvalues are again in accordance with those calculated by Roberts (1972) for the same model.

The stability of these two solutions of the nonlinear problem is examined by following the evolution of $P$ for different initial conditions and values of $C_\omega$. A surprising new situation in comparison with the foregoing models appears: For $C_\omega = 0.75$ the antisymmetric solution proves to be stable (Fig. 10a). However, for $C_\omega = 0.9$ our calculations reveal that neither of the two nonlinear solutions is stable (Fig. 10b, c). The time behaviour of $P$ exhibits a long period, which is about ten times the basic magnetic period. Within this long period, obviously, the magnetic field switches from a dominant symmetric state ($P > 0$) to a state where the antisymmetric part is dominating ($P < 0$) and vice versa. The trajectory in phase space of the system lies on a torus (Brandenburg et al., 1989). Then, for $C_\omega = 1$ the symmetric

![Fig. 10a](image1.png)

**Fig. 10a.** Evolution of $P$ for the oscillatory $z\omega$-dynamo model of Sect. 6.4 for $C_\omega = 0.75$. The symmetric solution ($P = +1$), which already exists, is unstable. Hence the stable antisymmetric one represents the final state.

![Fig. 10b](image2.png)

**Fig. 10b.** Same as Fig. 10a, but with $C_\omega = 0.9$. Now the antisymmetric solution is also unstable. The attractor is more complex: it shows in the extreme cases either dominant symmetry or antisymmetry. The long period of this parity variation is about ten times the basic magnetic period. Note that the high frequency wiggles are due to the energy, which oscillates with half the magnetic period.

![Fig. 10c](image3.png)

**Fig. 10c.** As Fig. 10b, but starting from $P = -0.9$. The same solution is reached from this initial condition too. The length of the bar in the upper part of the diagram is $2\pi(\Omega_\omega^{(0)} - \Omega_\omega^{(\text{lin})})$, where $\Omega_\omega^{(\text{lin})}$ are the nonlinear frequencies for the pure solutions. This time coincides with the long term period.
solution is stable and obviously the final state for arbitrary initial conditions (Fig. 10d).

From these calculations we may conclude that there is a value $C_s^*$ with the property that the antisymmetric solution is stable for $C_s < C_s^*$ and unstable for $C_s > C_s^*$. Furthermore there is a value $C_s^{**} > C_s^*$ such that the symmetric solution is unstable for $C_s < C_s^{**}$ but stable for $C_s > C_s^{**}$. For $C_s^* < C_s < C_s^{**}$ no stable nonlinear solution of pure parity exists (Fig. 10b, c).

One is tempted to apply the main result of these investigations to the Sun. It is well known that the poloidal field is not purely antisymmetric, e.g. it was observed that the polar caps reverse polarity at different times (e.g. Babcock, 1959), thus indicating a superposed symmetric (quadrupole) field. The field geometry calculated for $C_s = 0.9$ shows similar features, the relative magnitude of antisymmetric and symmetric part slowly changing over the long period. However, in the case of the Sun, one must take into account that the butterfly diagram, which represents the toroidal field and so the strong part of the solar magnetic field, does not show significant deviations from antisymmetry. In addition, Stenflo and Vogel (1986) have not found even parity field parts of the poloidal field with a definite period.

The process of change of symmetry lasts for many periods, as can be seen from Fig. 10e showing the toroidal field at the depth $r = 0.95$ in the $t - \theta$ plane (butterfly diagram). One clearly sees that the mean magnetic field near the equator grows from (nearly) zero to a maximum, when the preferred symmetric configuration is reached.

7. Conclusions

We have shown that, for all considered dynamo models, the kinematic growth rates of fields of different parities are asymptotically equal as the dynamo numbers increase. Hence it is clear that a selection of a field with a certain parity cannot in general be explained by consideration of kinematic growth rates.

We then considered the behaviour of some models taking into account a nonlinear back-reaction of the magnetic field of various forms. In general we confirm the finding of Krause and Meinel (1988) that the stability of the nonlinear solutions provides the parity selection, although here, because more complicated models are considered, the phenomena are more complex. We confirmed that the first excited solution (in the order of growing dynamo number) is, for a certain range, the only stable one and that the higher solutions are at first unstable. However, in two cases it was found that there is a certain value $C_s^*$ of the dynamo number, beyond which the second nonlinear solution becomes also stable (see Fig. 8e). The final state proved to be dependent on the initial conditions being, however, always of one
symmetry. No mixture of fields with different symmetries appeared.

The higher bifurcation points are of less physical interest since the solutions will not be found or the corresponding fields observed because they are unstable. In contrast $C_\omega^*$ is a quantity of interest since it characterizes the beginning of the range of the dynamo number where more than one stable nonlinear solution exists.

A new situation appears in our last example, where an oscillatory $\omega$-dynamo has been considered. For a constant $C_\omega$ and with increasing $C_\alpha$ the antisymmetric solution is at first stable. However, when $C_\alpha$ is increased more, it loses its stability, whilst the symmetric solution remains still unstable. No stable oscillating solution of a definite parity exists. Any initial field is attracted by a field with two periods: a long term oscillation between the unstable solutions of genuine symmetries superposed over the normal magnetic cycle. For even larger $C_\alpha$ the symmetric solution becomes stable. Obviously this model is worthy of further investigation.

The basic mechanism stabilizing or destabilizing different solutions is not well understood. To clarify this it therefore seems necessary to enlarge the sample of dynamo models with different stability behaviour. For example it would be important for application to the Sun to know under which conditions the oscillatory 40 mode is stable. Furthermore, the stability analysis of spherical dynamos has so far been restricted to axisymmetric analysis. So we do not know whether a solution, which is here proved to be stable, will respond similarly for non-axisymmetric disturbances. Results of Rädler and Wiedemann (1989) suggest that some of our solutions may possibly be unstable to non-axisymmetric perturbations.

Observations of the magnetic fields show more or less significant deviations from the basic symmetry, e.g. the inclination of the Earth's dipole or the sectorial structure of the Sun. Probably these deviations from symmetry have to be explained by models where the condition of exact symmetry of the internal structure and motions is relaxed from the beginning. This is, however, beyond the scope of this paper.

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