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Properties of mean field dynamos with non–axisymmetric α-effect

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Abstract. We investigate the influence of an azimuthally dependent α-effect on the properties of α^2 and $\alpha^2\Omega$-dynamos in spherical geometry, restricting ourselves to odd parity solutions in linear theory. For all our linear models we find an exponentially growing mode, consisting of locked axisymmetric and nonaxisymmetric parts. A strong nonaxismmetry in α substantially increases the linear growth rates at given dynamo number and can result in a marginal dynamo number that is significantly smaller than for an axisymmetric α with the same value of $\int \alpha dV$. We also report some exploratory nonlinear calculations and briefly discuss the relevance of our results to galactic dynamos and to stars.

Key words: Galaxies: magnetic fields – hydromagnetics – turbulence – dynamo theory

1. Introduction

Axisymmetric linear mean field dynamo theory is approaching its quarter century – a respectable age! It has spawned a plethora of papers, initially mainly concerned with the solar field, but more recently applied also to flattened objects (e.g. galaxies). Numerous mechanisms leading to nonlinearities have also been investigated. Although nonlinear behaviour is undoubtably crucial for understanding stellar dynamos, where growth times are very much less than the ages of the objects, its importance is perhaps a little less clear for some galactic dynamos with relatively slower growth rates (e.g. Ruzmaikin et al., 1988)

One of the outstanding problems of galactic dynamo theory is to explain the diversity of the observed field structures. Some galaxies have axisymmetric spiral fields (ASS), some bisymmetric spirals (BSS), and others have no well-defined morphology. In terms of dynamo theory, the ASS fields correspond to axisymmetric ($m = 0$) modes and the BSS fields to nonaxisymmetric ($m = 1$) modes. It has proved difficult to produce "flat" (i.e. disk-like) dynamo models with axisymmetric α-effect in which nonaxisymmetric modes are preferentially excited (see, e.g., Meinel et al. 1990). This has led to the introduction of the concept of a nonaxisymmetric α-effect (e.g. Chiba & Tosa, 1989; Mestel & Subramanian, 1991). One physical motivation is that zones of vigorous star formation in spiral arms may create regions of stronger than average turbulence in the interstellar medium, and correspondingly an enhanced nonaxisymmetric α-coefficient.

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In this paper we make a very preliminary and idealized study of the effects of such a nonaxisymmetric α-effect on the basic properties of mean field dynamos. In order to get a clear picture of the effects of such a mechanism we have restricted ourselves to calculating spherical dynamos. Our formalism in principle can readily be adapted to perform calculations of the "embedded disk" type used by Elstner et al. (1990) and Moss & Tuominen (1990), although providing adequate spatial resolution would make the computations significantly more expensive. We feel that our results provide some insight into the general properties of such mean field dynamos that may, with some modification, be applicable to the problem of the galactic field.

2. Method

We solve the linear mean field dynamo equation

$$\partial \boldsymbol{B}/\partial t = \mathrm{curl}\,(\alpha \boldsymbol{B} + \boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \quad \text{with} \tag{1}$$

$$\alpha = \alpha_0 \cos\theta (1 + \delta \cos\lambda), \quad 0 \le \delta \le 1. \tag{2}$$

We adopted the simplest differential rotation law,

$$\Omega = \Delta\Omega x, \tag{3}$$

where $x = r/R$ is the fractional radius and α_0 and $\Delta\Omega$ are constants. Note that $\int \alpha dV$ is independent of δ. r, θ, λ are spherical polar coordinates.

$$C_\alpha = \alpha_0 R/\eta, \quad C_\Omega = \Delta\Omega R^2/\eta. \tag{4}$$

are the usual dynamo parameters, and we also define the overall parity parameter, $P = [E^{(S)} - E^{(A)}]/[E^{(S)} + E^{(A)}]$, and symmetry parameter $M = 1 - E^{(0)}/E_{\rm tot}$. $E^{(A)}$, $E^{(S)}$, $E^{(0)}$, and $E_{\rm tot}$ are respectively the energies in the parts of the field anti- symmetric and symmetric with respect to the plane $\theta = \pi/2$, the energy of the axisymmetric field component, and the total field energy.

Although our calculations are linear, we found it convenient to use a slightly modified version of the nonlinear nonaxisymmetric dynamo code described in Moss et al. (1991), and determined the linear modes and linear growth rates for given values of C_α and C_Ω by integrating Eq. (1) until the field strength increased exponentially with time and the field structure had become invariant. The initial fields could be arbitrarily chosen. As we are for now interested largely in elucidating the general properties of dynamos with nonaxisymmetric α-effect, rather than performing a comprehensive survey of a particular dynamo model, our linear calculations are restricted to fields of odd parity, $P = -1$.

For the majority of our computations we used 31 radial grid points, uniformly distributed over $0 \le r \le R$ and 31 points uniformly distributed over $0 \le \theta \le \pi/2$, and included explicitly the $m = 0$ to $m = 3$ modes. We checked that decreasing the mesh size or increasing the number of nonaxisymmetric modes included did not significantly alter our results. Time steps, in units of the global diffusion time, were between 10^{-4} and 2.5×10^{-5}.

3. Results

3.1. α^2-dynamos

We have the standard results (with resolution 31×31 grid points) that with $\delta = 0$ the A0 mode is excited at $C_\alpha = 7.64$ (and the S0 mode at $C_\alpha = 7.82$). The $m = 1$ modes are excited at similar values of C_α: A1 at 8.0, (S1 at 7.7). We took $C_\alpha = 10$, a clearly supercritical value, and investigated the nature of the fastest growing linear mode as δ was increased from zero.

We started our computations from a configuration with either $M = 0$ or $M = 1$, and varied δ in the range (0,1). For $\delta > 0$ the field eventually grew exponentially with structure independent of the initial configuration. If M_f is the value of M for this eigenmode, then M_f increases with δ, approaching but not attaining, the value of unity as δ increases to 1, see Fig. 1. A noteworthy feature of these calculations is that the growth rates increase quite rapidly with δ, see Table 1. In this table M_f the final, steady value of M, ν the growth rate of the eigenmode and e_1, e_2 and e_3 are the fractions of the energy in the $m = 1,2,3$ parts of the field respectively. For small values of δ ($\lesssim 0.05$), calculations starting with $M = 1$ did appear to maintain values of M near to unity for several diffusion times, whereas for initial $M \ge 0.10$ the decrease in M was immediate.

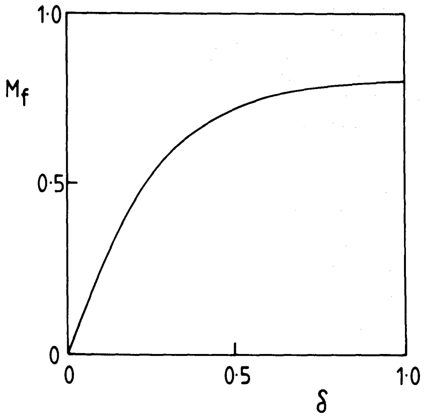

Fig. 1. Global symmetry parameter M_f against δ for calculations with $C_\alpha = 10$, $C_\omega = 0$

3.2. $\alpha^2\Omega$-dynamos

It is well known that for large values of C_Ω nonaxisymmetric modes are strongly inhibited (e.g. Rädler, 1986). We thus chose a relatively modest value, $C_\Omega = -300$. The corresponding C_α values for excitation of the A0, S0, A1, S1 modes are then respectively 9.5, 10.4, 15.2, 15.2 when $\delta = 0$. We first investigated the case $C_\alpha = 11$, chosen so that with $\delta = 0$ the axisymmetric modes are excited but not the nonaxisymmetric. With an A0 or A1 type initial field we found the results given in Table 2. In contrast to the previous results (Sect. 3.1), the final stable mode has only a small nonaxisymmetric component (small M value) even for $\delta = 1$, and the growth rates only increase slightly with δ. There appears to be no mode with large values of M. The fraction of the energy in the $m > 1$ part of the eigenmodes never exceeds 0.001.

We then put $C_\alpha = 16$, so that both $m = 0$ and $m = 1$ modes are excited in linear theory with $\delta = 0$. For $\delta = 0$ the axisymmetric modes dominate. As δ increases, the behaviour is similar to that found when $C_\alpha = 10$, $C_\Omega = 0$: the growth rates increase markedly and the dominant mode becomes increasingly nonaxisymmetric, see Table 3. When $\delta = 1$, approximately 5% of the field energy is in the $m > 1$ modes. Thus e_2 and e_3 are very small and $e_1 \approx M_f$.

3.3. Depression of $C_{\alpha,\mathrm{crit}}$

In the light of the above result, that increasing δ increases the linear growth rates at fixed C_α, we kept $\delta = 1$, $C_\Omega = 0$, and determined the corresponding marginal $C_\alpha = 5.5$ for the mixed

Table 1. Summary of results for $C_\alpha = 10$, $C_\Omega = 0$. For the first entry the initial configuration has $M = 1$, otherwise the final field configuration is independent of the initial M value

δ	M_f	ν	e_1	e_2	e_3
0.00	1.00	18.0	1.00	0.0	0.0
0.05	0.06	18.3	0.059	$5.6\,10^{-4}$	$1.8\,10^{-6}$
0.10	0.20	20.3	0.19	$7.3\,10^{-3}$	$9.1\,10^{-5}$
0.20	0.47	23.9	0.39	$7.6\,10^{-2}$	$3.3\,10^{-3}$
0.60	0.76	54.4	0.49	0.23	0.04
1.0	0.80	100	0.47	0.27	0.06

Table 2. Summary of results for $C_\alpha = 11$, $C_\Omega = -300$

δ	M_f	ν
0.025	0.00002	8.0
0.10	0.0003	8.0
0.25	0.0017	8.0
1.0	0.030	9.5

Table 3. Summary of results for $C_\alpha = 16$, $C_\Omega = -300$. The asterisk denotes an approximate value, calculation not run long enough to determine the value exactly

δ	M_f	ν
0.001	$5\,10^{-7}$*	45
0.02	0.0002*	45
0.10	0.005	52.2
0.25	0.03	57.5
1.0	0.37	117

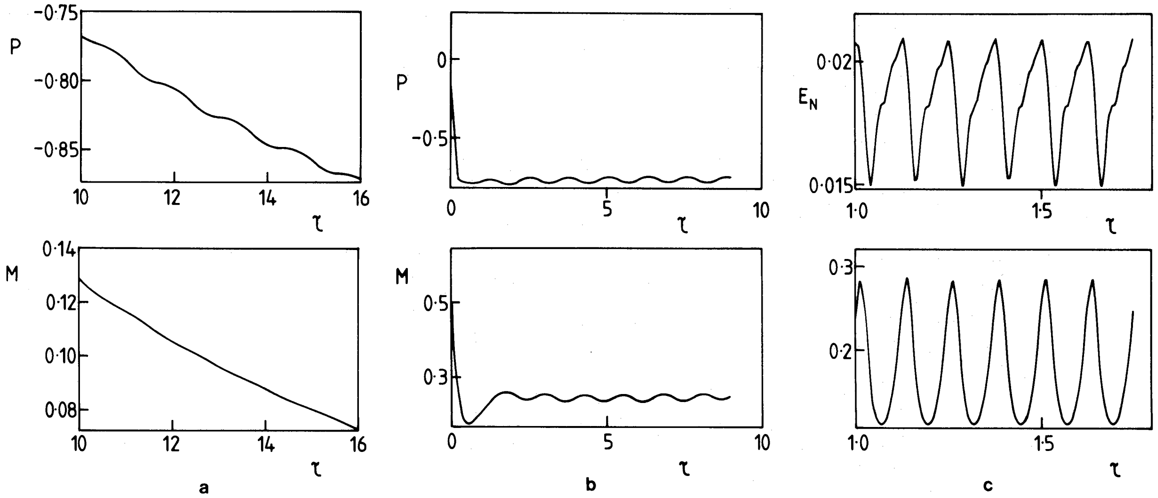

Fig. 2. Variations of M and P with τ for calculations with a) $C_\alpha = 10$, $C_\Omega = 0$, $\delta = 0.05$, b) $C_\alpha = 10$, $C_\Omega = 0$, $\delta = 0.20$; c) variation of $E^{(N)} = E_{tot} - E^{(0)}$ and M when $C_\alpha = 16$, $C_\Omega = -300$, $\delta = 1.0$

odd parity mode, which has $M = 0.84$. This can be compared with the standard value $C_{\alpha,\mathrm{crit}} = 7.64$ when $\delta = 0$, $M = 0$.

We made a similar investigation for the case $C_\Omega = -300$. Our code, with our standard resolution, gave a marginal C_α value for the A0 mode with $\delta = 0$ of approximately 9.57. With $\delta = 1$ the marginal value for the mixed odd parity mode (M ca. 0.02) became approximately 9.2.

3.4. Nonlinear calculations

In a spirit of investigation we performed some nonlinear calculations with α multiplied by a factor $1/[1 + \boldsymbol{B}^2(r, \theta, \lambda)]$. For these calculations we did not select any particular parity for our fields, and our standard computational grid had 61 points distributed uniformly over the range $0 \leq \theta \leq \pi$. These are summarized in Table 4. Starting with an initial field configuration of mixed parity, in which $M = 0.5$, for $C_\alpha = 10$, $C_\Omega = 0$ and $\delta = 0.2$ we found what appeared to be a mixed parity limit cycle with $0.23 < M < 0.25$ and $0.74 < P < 0.76$, although we did not run the computation long enough to be absolutely sure that we had precisely determined the final state, see Fig. 2. When δ was reduced to 0.05, M quickly became very small. In the latter case we also verified that there was no stable solution in the vicinity of $P = -1$, $M = 1.0$: calculations initiated in this region evolve towards small M both if only strictly odd parity fields are considered or if initially P has a value close to but not exactly -1. When C_α was increased to 15 with $\delta = 0.2$, the solution was evolving slowly to a configuration with small M when the computation was halted. In general, these α^2 dynamos evolved very slowly, and it was computationally too expensive to follow them for long enough to determine fully the final configurations. It is plausible, for example, that the models with $C_\alpha = 10$, $\delta = 0.05$ and $C_\alpha = 15$ would eventually evolve to $P = -1$.

With $C_\alpha = 16$, $C_\Omega = -300$, $\delta = 1.0$ we found a limit cycle with M varying between values of 0.01 and 0.03 approximately. Some details of these nonlinear solutions are given in Fig. 2.

Table 4. Summary of nonlinear calculations. (1) Computation halted at $\tau = 16$ when M and P still decreasing very slowly. (2) Values at $\tau = 9.0$. P and M may eventually become steady. (3) Computation halted at $\tau = 7$ when M and P still decreasing very slowly

C_α	C_Ω	δ	M_f	P_f	Comments
10.0	0	0.05	< 0.07	$< -.87$	(1)
10.0	0	0.20	$0.23 - 0.25$	$0.74 - 0.76$	(2)
15.0	0	0.20	< 0.16	$< -.75$	(3)
15.0	-300	0.20	0.001	-1.0	
16.0	-300	1.0	0.011...0.029	-1.0	

4. Discussion

The results presented in Sect. 3.2, although very limited in nature, do not present any evidence for the existence of configurations in which the presence of a nonaxisymmetric α-effect can excite a dominant linear nonaxisymmetric dynamo mode where otherwise only $m = 0$ modes would be excited in linear theory. In such cases the resulting mixed modes are just slight perturbations to the axisymmetric modes. Although the results presented in Sections 3.1 and 3.2 are restricted to purely odd parity fields, there is no reason to expect their general nature to change for fields restricted to be of strictly even parity. This expectation is supported by a trial computation with $P = +1$.

When axisymmetric and nonaxisymmetric modes are both clearly excited according to linear theory, then a nonaxisymmetric α-effect can produce a linear eigenmode of fastest growth rate that has a mixed parity and departs substantially from axisymmetry, see Table 1. For our models M does not (and indeed cannot) attain unity – there must inevitably be both $m = 0$ and $m = 1$ (and $m > 1$) field components present.

However when the dynamo parameters are chosen to be only slightly supercritical for the $m = 1$ modes, but substantially supercritical for the $m = 0$, then once again even a strong nonaxisymmetry in α produces only a weakly nonaxisymmetric eigenmode – see Table 3. We did perform a couple of rather speculative calculations with $C_\Omega = -300$ and $C_\alpha > 16$. These were of limited accuracy because of the very rapid field growth at these highly supercritical parameters but they did suggest that as C_α

increases the M value of the first eigenmode increases significantly at given δ, so that the above comments may only be relevant in a fairly restricted parameter range near to the marginal C_α for the harder-to-excite modes. We can speculate that the behaviour described in Sect. 3.1 may be normal when both axisymmetric and nonaxisymmetric modes are excited in linear theory with axisymmetric α, with growth rates that are not too dissimilar. If only, e.g., axisymmetric modes are excited, then the behaviour described in Sect. 3.2 for the case $C_\alpha = 11$, $C_\Omega = -300$, may be typical.

Some insight into these results in the linear theory can be obtained by examining the dynamo equation (1). Suppose that at given C_α, for α independent of λ, Eq. (1) has solutions

$$\boldsymbol{B}_0 = \boldsymbol{b}_0(\boldsymbol{r})e^{\nu_0 t}, \quad \boldsymbol{B}_1 = \boldsymbol{b}_1(\boldsymbol{r})e^{\nu_1 t + i\lambda}. \tag{5}$$

Write

$$\mathscr{D} = \partial/\partial t + \mathrm{curl}\,(\eta \mathrm{curl} - \mathrm{curl}\,(\alpha_0), \tag{6}$$

and suppose that $\alpha = \alpha_0 + \alpha_1$, where α_0 is independent of λ and $\alpha_1 \sim e^{i\lambda}$. Consider $|\alpha_1| = O(\delta)\ |\alpha_0|$, where δ is small and put $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1 + \ldots$. Then

$$\mathscr{D}\boldsymbol{B}_0 \sim \mathrm{curl}\,(\alpha_1 \boldsymbol{B}_1) \quad \text{and} \tag{7}$$

$$\mathscr{D}\boldsymbol{B}_1 \sim \mathrm{curl}\,(\alpha_1 \boldsymbol{B}_0), \tag{8}$$

Now assume that we start with a $m = 0$ field so that, initially at least, $|\boldsymbol{B}_1| \ll |\boldsymbol{B}_0|$. Then the term on the rhs of (7) can be neglected to first order, and $\boldsymbol{B}_0$ grows with growth rate ν_0. In (8) the term $\alpha_1\boldsymbol{B}_0$ acts as a source for the $\boldsymbol{B}_1$ field which then grows. If $\nu_0 \gg \nu_1$, then $\boldsymbol{B}_1$ is slaved to $\boldsymbol{B}_0$ and $|\boldsymbol{B}_1| = O(\delta)\ |\boldsymbol{B}_0|$. If $\nu_0 \gtrsim \nu_1$, again we expect $|\boldsymbol{B}_1| = O(\delta)\ |\boldsymbol{B}_0|$.

Now start with a $m = 1$ field, so that initially $|\boldsymbol{B}_0| \ll |\boldsymbol{B}_1|$. The zeroth order solution of (8) is an exponential growth of $\boldsymbol{B}_1$ provided $\nu_1 > 0$. The lowest order terms in (7) are of 1st order and all must be retained. The term curl $(\alpha_1\boldsymbol{B}_1)$ acts as a source for the field $\boldsymbol{B}_0$, which grows with growth rate $\max(\nu_0, \nu_1)$. If $\nu_1 \gtrsim \nu_0$ we can expect $|\boldsymbol{B}_1| = O(\delta)\ |\boldsymbol{B}_0|$. If, in contrast, $\nu_0 \gg \nu_1$, then $\boldsymbol{B}_1$ is slaved to $\boldsymbol{B}_0$ with $|\boldsymbol{B}_1| = O(\delta)\ |\boldsymbol{B}_0|$. From this it appears that we might expect two distinct modes, with M near 0 and 1 respectively, for small values of δ if ν_0 and ν_1 are of similar magnitude (cf. remark at the end of Sect.3.1). Otherwise slaving will occur and the only mode will be a perturbation to the mode with fastest growth rate.

This seems to be consistent with the behaviour for small δ noted in Sect. 3.1 and 3.2, except that we always find a unique value of M_f, even in Sect 3.1 where ν_0 and ν_1 are comparable. We speculate that this may be because the solutions with M_f near unity are unstable in linear theory as well as when nonlinearities are included (Sect. 3.4). The behaviour of solutions with initial $M = 1$ and $\delta \lesssim 0.05$ described in Sect. 3.1 may then be a consequence of such an instability.

The other case is when, at given C_α, $\nu_0 > 0$, $\nu_1 < 0$. Then Eq. (7) is approximated for all δ by $\mathscr{D}\boldsymbol{B}_0 = 0$, and $\boldsymbol{B}_1$ is slaved to $\boldsymbol{B}_0$ by the source term curl $(\alpha_1\boldsymbol{B}_0)$ in Eq. (8). Thus we expect $|\boldsymbol{B}_1| = O(\delta)\ |\boldsymbol{B}_0|$ – this is the only possibility. This appears to be the situation when $C_\alpha = 11$, $C_\Omega = -300$.

A slightly unexpected result is that for the same $\int \alpha dV$, a nonaxisymmetric α (Eq. (2)) is more effective at exciting a dynamo than an axisymmetric α, and mixed parity eigenmodes grow for values of C_α at which purely axisymmetric fields decay. The effect is clearly much more dramatic when the marginal C_α values for the $m = 0$ and $m = 1$ modes are close together (Sect. 3.1) than when they are well separated (Sect. 3.2).

The introduction of a simple nonlinearity can sometimes produce an increased degree of nonaxisymmetry, but not always (see the solution with $C_\alpha = 16$, $C_\Omega = -300$). In the $C_\alpha = 16$, $C_\Omega = -300$ case, at least, the results are of a generally similar nature to those of Moss et al. (1991), who investigated nonlinear nonaxisymmetric dynamos with axisymmetric α-effect, with less simple α and Ω distributions than considered here. The inclusion of such a nonlinearity also enables us to investigate the stability of our odd parity solutions for the α^2 dynamo, and shows that the solution branch starting from $M = 1$, $P = -1$ is unstable.

What do these results mean in the context of galactic field generation? Our models have too many limitations, not least the absence of any "flattening", to enable any but the most general comments to be made. Our differential rotation is distinctly "non-galactic" in form and our α-effect is nonaxisymmetric but not of "spiral" form. Also anisotropies in α (i.e. a tensorial form) should perhaps be considered (e.g. Rüdiger, 1990). However they do suggest that a "spiral α", presumably related to processes occuring in galactic spiral arms, might produce some interesting effects – possibly exciting a dynamo at a lower value of a dynamo number than might otherwise be expected and also, in some cases, resulting in a field that is clearly of neither purely $m = 0$ nor $m = 1$ type.

It is even possible that nonaxisymmetric α effects may influence some stellar dynamos. Possibilities include situations where there are strong tidal effects in close binary systems or if the effects of giant cells dominate. In this context the nonaxisymmetric structure inferred on RS CVn stars (e.g. Zeilik, 1991) and FK Comae stars (e.g. Jetsu et al., 1990; Piskunov et al., 1990) could be relevant.

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