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# ON THE BIFURCATION PHENOMENA OF THE KURAMOTO–SIVASHINSKY EQUATION

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Bifurcation phenomena of the Kuramoto–Sivashinsky equation have been studied numerically. The solutions considered are restricted to the invariant subspace of odd functions. One possible route to chaos via a period-doubling cascade is investigated in detail: The four-modal steady-state loses its stability through a Hopf bifurcation and a branch of periodic motions is created. After a symmetry breaking the periodic solution undergoes a period-doubling cascade which ends up in two antisymmetric chaotic attractors. A merging of these antisymmetric attractors to a symmetric one is observed. The chaotic branch depending on the bifurcation parameter is characterized by the values of the Lyapunov exponents. Periodic windows within the chaotic region are also detected. Finally, a further increase of the bifurcation parameter leads to a transition from the attractor into transient chaos.

## 1. Introduction

It is well known and frequently published in the literature that the Kuramoto–Sivashinsky equation (KS), in spite of its simple structure, possesses a very complex solution behavior.

The equation was first derived independently by Kuramoto & Tsuzuki [1975, 1976], Kuramoto [1977] and Sivashinsky [1977, 1980], to describe certain reaction-diffusion systems and the dynamics of two-dimensional flame fronts, respectively. Numerical experiments by Kuramoto gave hints that the KS equation possesses turbulent solutions. After this discovery the equation became a topic of intensive research to classify the solutions and to describe their properties [Michelson, 1986; Hyman & Nicolaenko, 1986, Hyman *et al.*, 1986; Greene & Kim, 1988].

The bifurcation theory provides a powerful tool for discovering several solution branches, and for detecting chaotic regions more systematically than is possible using only forward integration in time. In a couple of recent papers the steady-state bifurcation structure of the KS equation using higher mode truncation has been determined [Foias *et al.*, 1988; Jolly *et al.*, 1990; Kevrekidis *et al.*, 1990]. Starting from these results the purpose of this letter is to study the appearance of a chaotic attractor and its decay. For the investigation we used the software system CANDYS/QA which calculates branches of steady-states and of periodic solutions including all bifurcations of codimension 1 [Feudel & Jansen, 1992]. We consider the KS equation in the form

$$\frac{\partial u}{\partial t} + 4\frac{\partial^4 u}{\partial x^4} + \alpha\left(\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\right) = 0\,, \qquad (1)$$

subject to periodic boundary conditions $u(x, t) = u(x + 2\pi, t)$ on the $x$-interval $0 \le x \le 2\pi$. There exists a unique solution $u(x, t) \in H^4_{\text{per}}(0, 2\pi)$ for this initial-boundary value problem in the subspace of $2\pi$-periodic functions in the Sobolev space $H^4(0, 2\pi)$ [Nicolaenko & Scheurer, 1984]. As in Jolly *et al.* [1990] we restrict the solutions to the invariant subspace of odd functions expressed by the condition $u(x, t) = -u(2\pi - x, t)$. One reason for this restriction is to avoid degenerations of the eigenvalues of the linearized equations and to apply the classification scheme for codimension 1 bifurcations.

In the original form all coefficients of the KS equation are unity and the initial-boundary value problem is considered on an interval of length $L$. A rescaling transformation of space and time leads to Eq. (1) where the new bifurcation parameter $\alpha$ is connected with the length $L$ of the original equation by $\alpha = L^2/\pi^2$ [Jolly *et al.*, 1990]. For the bifurcation analysis this normalized form is used here. We employ a 24 mode truncation to approximate the KS equation by means of the eigenfunctions of the linearized operator $u_k(x) = \sin kx$. The time evolution of the components of the series expansion is determined by the following ODE

$$\dot{x}_k = (-4k^4 + \alpha k^2)x_k - \alpha\beta_k^m\,, \quad 1 \leq k \leq m\,, \qquad (2)$$

where

$$\beta_k^m = \frac{1}{2}\sum_{j=1}^{m} jx_j[x_{k+j} + \text{sgn}(k-j)x_{|k-j|}]\,. \qquad (3)$$

The index $m$ represents the number of modes taken into account in the truncation and we set $x_j = 0$ if $j < 1$ and $j > m$, respectively. This particular form of Eq. (2) has already been derived by Jolly *et al.* [1990].

## 2. The Emergence and the Decay of a Chaotic Branch

Greene & Kim [1988] investigated the steady solutions and gave an overview of their bifurcation structure. Jolly *et al.* [1990] calculated the complete bifurcation diagram of steady-states considering only odd functions as solution space for the bifurcation parameter $\alpha$ in the range $0 \leq \alpha \leq 70$. Starting from these results we extended this bifurcation diagram to the larger interval $0 \leq \alpha \leq 150$ to present the steady solution branches in the neighborhood of the investigated chaotic attractor (cf. Fig. 1). Stationary states branching off the trivial solution at $\alpha = 4k^2$ ($k \in \mathbb{N}$) are called $k$-modal steady-states. In particular, the first and second modal branches are known as the unimodal and bimodal branches, respectively. Besides these primary bifurcations a lot of secondary bifurcations branching off the $k$-modal steady-states occur. In Fig. 1 the $L^2$-norm $|u|$ depending on the bifurcation parameter $\alpha$ is given.

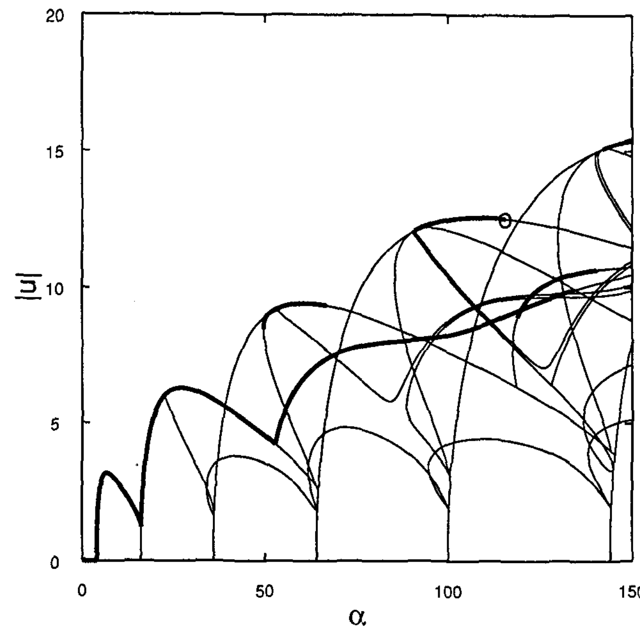


Fig. 1. Complete bifurcation diagram of the steady-states. The "o" denotes the Hopf point whose resulting periodic motion ends up in chaos.

Each line in the figure represents the projection of two branches having the same norm but differing in the sign of certain modes. Additionally, these two branches may differ in their stability and their bifurcation behavior. However, for the sake of simplicity we have not distinguished between the cases where only one of the two branches of the projection is stable and where both of them are stable. Such differentiated stability behavior concerns only the bimodal branch whose bifurcations are described in detail by Jolly *et al.* [1990]. More precisely, branches where at least one of the steady-states is stable are drawn by bold lines.

Let us briefly discuss the bifurcation diagram shown in Fig. 1. In general, the $k$-modal steady-states have $k - 1$ positive real eigenvalues near to their critical points at $\alpha = 4k^2$, and they become stable after a sequence of $k - 1$ pitchfork bifurcations. Eventually, the $k$-modal branches for $k \geq 3$ undergo a Hopf bifurcation and lose their stability.

At these Hopf points periodic solutions arise whose branches are omitted in Fig. 1.

In the following we study the bifurcation behavior of only one particular periodic solution that results in a chaotic branch. We continued the periodic solution which is created by a Hopf bifurcation at $\alpha = 115.511$ of the negative four-modal branch. This special Hopf point is denoted by "∘" in Fig. 1.

At first the periodic solution shows a symmetry in the projection onto the plane spanned by the first two modes with respect to the origin. It is known that such symmetry conditions can suppress a period-doubling bifurcation [Swift & Wiesenfeld, 1984; Brown *et al.*, 1991]. Only after a symmetry breaking at $\alpha \approx 131.92$ do both antisymmetric solutions undergo a period-doubling cascade. The first three period doublings have been observed at the values of $\alpha$ given in Table 1. A numerical estimation of further period doublings was not possible. Eventually, a pair of two antisymmetric attractors is obtained firstly at $\alpha = 133.46$.

Table 1. Period-doubling bifurcations on the route to chaos.

| | $\alpha$ |
|---|---|
| period-doubling $T \to 2T$ | 133.110 |
| period-doubling $2T \to 4T$ | 133.390 |
| period-doubling $4T \to 8T$ | 133.452 |

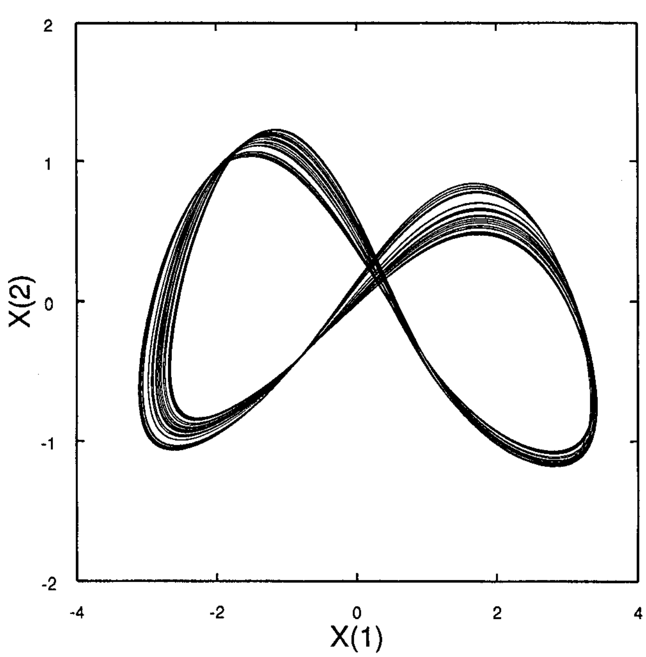


Fig. 2. One of the two antisymmetric attractors shown as a projection onto the $x_1$–$x_2$ plane for $\alpha = 133.6$.

To check the sensitivity with respect to small changes in the initial conditions we calculated the first four Lyapunov exponents. The largest one has been found to be positive which indicates chaotic behavior for this value of $\alpha$. For the computation of the Lyapunov exponents the algorithm developed by Shimada & Nagashima [1979] has been used. As an example of such a chaotic solution the projection of one of the two antisymmetric attractors onto the first two modes is shown in Fig. 2. At $\alpha \approx 133.77$ a merging of both antisymmetric attractors into a symmetric one occurs, which can be regarded as a symmetry increasing bifurcation.

To discuss the chaotic properties the projection of the chaotic attractor and the power spectrum of its norm for fully developed chaos at $\alpha = 135.0$ is presented in Figs. 3 and 4. In Fig. 3 the attractor is shown after the merging of both antisymmetric attractors where it is again symmetric with respect to the origin of the coordinate system. The power spectrum in Fig. 4 is not purely continuous and contains a large number of peaks according to periodic intervals in the time series of the norm. With increasing values of the bifurcation parameter the attractor occupies a progressively larger region in phase space until it decays into transient chaos at $\alpha \approx 137.0$.

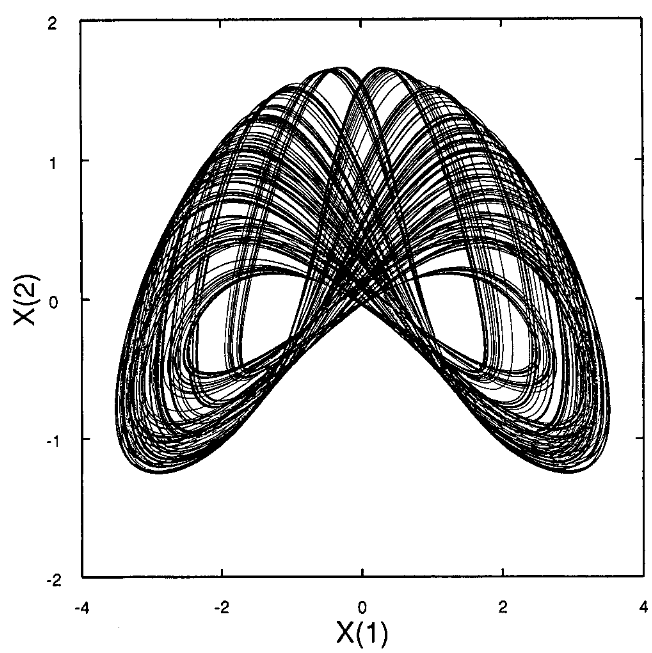


Fig. 3. The attractor after merging of the two antisymmetric attractors shown as a projection onto the $x_1$–$x_2$ plane for $\alpha = 135.0$.

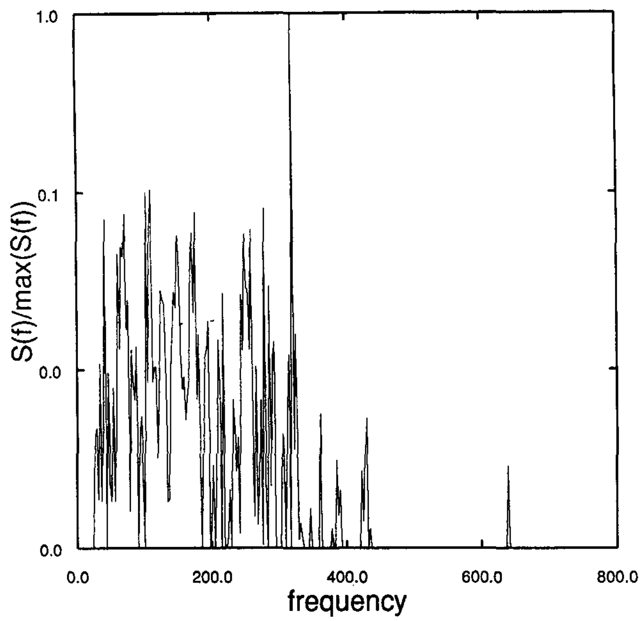


Fig. 4. Power spectrum of $|u|$ for $\alpha = 135.0$.

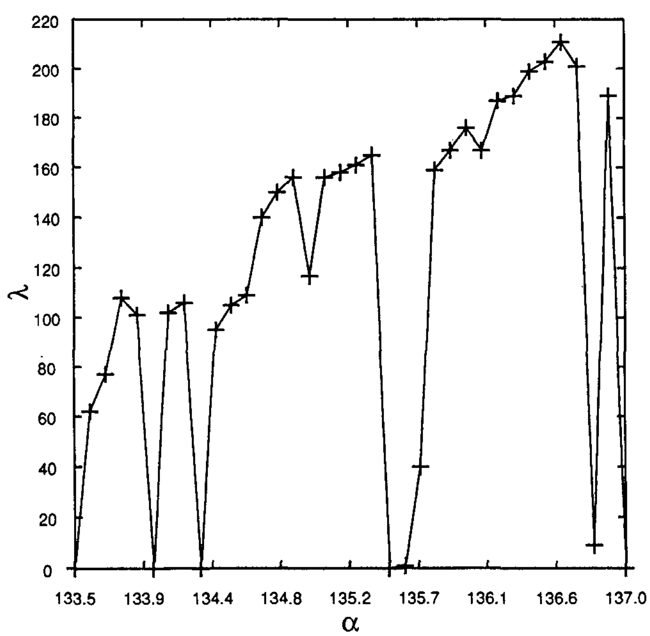


Fig. 5. The largest Lyapunov exponent as a function of $\alpha$.

To characterize the chaotic branch the first four Lyapunov exponents are calculated as a function of the bifurcation parameter within the interval $133.5 \leq \alpha \leq 137.0$. The largest Lyapunov exponent as a function of $\alpha$ is shown in Fig. 5. We recall that firstly the chaotic region was observed at $\alpha = 133.46$. But we consider here only the chaotic branch starting from $\alpha = 133.50$ where the first periodic window of period-6 is also located. The Lyapunov exponents are calculated with equidistant step size $\Delta\alpha = 0.1$ within the given interval and the corresponding points are labeled by crosses. We have connected the calculated points by a continuous line to emphasize the similarity between the structure of this chaotic branch and the corresponding diagram of the logistic map [Collet & Eckmann, 1980]. The Lyapunov exponent increases with a nearly constant slope interrupted by several periodic windows. In principle there is an infinite number of such periodic windows, but we only detected the largest of them:

— period-6: $\alpha = 133.5$
— period-5: $\alpha = 134.0$
— period-7: $\alpha = 134.3$
— period-3: $\alpha = 135.5$

A window exists for a certain interval of the bifurcation parameter. But because of the small length of this interval only one value of $\alpha$ indicating the window is given above. The windows arise in the same order as it is described in the book by Collet & Eckmann [1980]. That the window of period-3 is noticeably wider than the other windows is also known from the logistic map.

Using the formula of Kaplan & Yorke [1979] we estimated the Lyapunov dimension of the attractor. It is nearly constant for the chaotic branch and varies weakly between 2.09 and 2.17.

At a critical parameter value $\alpha_c = 137.0$ the largest Lyapunov exponent becomes again zero and a new region of transient chaos occurs for $\alpha > \alpha_c$. A typical trajectory for the transient chaos is presented in Fig. 6. It looks like the chaotic attractor for a long time. At random instants the trajectory leaves the vicinity of the former attractor, surrounds one of two stable symmetric cycles, comes back to the neighborhood of the former chaotic attractor and later surrounds again one of the cycles. After a possibly long, lifetime the chaotic transient escapes and approaches one of the stable cycles located symmetrically with respect to the origin of the coordinate system. Both stable periodic solutions exist already for $\alpha < \alpha_c$ but in this parameter region the basins of attraction of the cycles and of the chaotic attractor are well separated. For $\alpha = \alpha_c$ the chaotic attractor collides with its basin boundary and the trajectory can escape through a gap created. Such a phenomenon is called a crisis [Grebogi *et al.*, 1987].

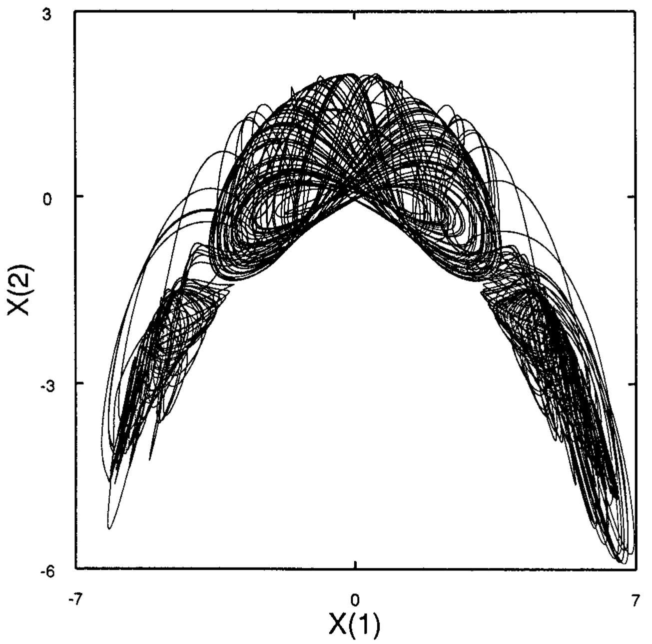


Fig. 6. The trajectory of the transient chaotic solution projected onto the $x_1$–$x_2$ plane for $\alpha = 137.1$.

## 3. Summary

The rich dynamical behavior of the KS equation is studied and one particular periodic solution that leads to chaos via a classical period doubling cascade is traced. The resulting chaotic branch shows interesting properties such as a merging of two antisymmetric attractors into a symmetric one and the transition to transient chaos, respectively. The chaotic attractor is characterized by the calculation of the largest Lyapunov exponent depending on the bifurcation parameter. Within the chaotic branch several periodic windows were detected and the corresponding figure (cf. Fig. 5) reveals the typical structure for a chaotic region.

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