THE ASTROPHYSICAL JOURNAL, 427:L91–L94, 1994 June 1

THE FORMATION OF SHARP STRUCTURES BY AMBIPOLAR DIFFUSION

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Received 1994 March 1; accepted 1994 March 21

ABSTRACT

The effect of ambipolar diffusion is investigated using simple numerical models. Examples are shown where sharp structures develop around magnetic nulls. In contrast to the case of ordinary diffusion, the magnetic field topology is conserved by ambipolar diffusion. This is demonstrated in an example where differential rotation winds up an initially uniform magnetic field and brings oppositely oriented field lines close together. It is argued that ambipolar diffusion produces structures of scales small enough for reconnection to occur.

Subject headings: galaxies: magnetic fields — ISM: magnetic fields — MHD

1. INTRODUCTION

Large-scale magnetic fields have been detected in many spiral galaxies (e.g., Beck 1993), and there are strong arguments that these fields are generated from a weak seed magnetic field via a turbulent hydromagnetic dynamo (e.g., Ruzmaikin, Shukurov, & Sokoloff 1988). The possibility of magnetic field reconnection is an essential part of such a theory. However, since the magnetic diffusivity is small (or the magnetic Reynolds number large), reconnection can only occur at small scales. In principle, the turbulent motions of the interstellar gas can mix the magnetic field to produce magnetic fluctuations at scales small enough so that the field is able to reconnect. This process leads to a much bigger *turbulent* magnetic diffusivity, which describes the diffusion of the large-scale magnetic field.

In galaxies the energy density of the magnetic field is comparable to the energy density of the turbulent motions, and this makes mixing less effective. This generally reduces the turbulent magnetic diffusivity (Roberts & Soward 1975), and it has even been suggested that at least in two dimensions the turbulent diffusion is completely blocked under such circumstances (Cattaneo & Vainshtein 1991). However, this does not necessarily occur in three dimensions, where a strong magnetic field can constrain the flow to planes perpendicular to the field lines (Krause & Rüdiger 1975). Furthermore, Kulsrud & Anderson (1992) have argued that in a weakly ionized gas like much of the interstellar medium in the galactic disk, ambipolar diffusion—or ion-neutral drift—can suppress the growth of magnetic structures at small scales. In response to some of this work, Parker (1992) proposed a dynamo model for the galaxy in which cosmic-ray pressure inflates the field into bubbles, which are then reconnected at their footpoints in the galactic disk. Thus, this model too invokes reconnection, but in a nontraditional manner.

In this *Letter* we show that small-scale magnetic structures can form in a weakly ionized gas, and that their formation is mediated by ambipolar diffusion. Its effect grows nonlinearly with increasing magnetic field strength, making the formation of sharp structures even more pronounced for strong magnetic fields. Therefore one must expect that a reduction of the turbulent magnetic diffusion for strong magnetic fields is compensated by the nonlinearity of ambipolar diffusion.

At first glance, the ability of ambipolar diffusion to produce sharp structures is somewhat surprising, because diffusion usually makes sharp gradients smoother (however, see Parker 1979 and Low 1982 for discussions of problems in which the constraint of magnetostatic equilibrium makes the ohmic diffusion problem effectively nonlinear and sharp structures can form). Ambipolar diffusion is highly anisotropic and nonuniform, and the name diffusion may therefore be misleading. Its main effect is to provide an additional term $\boldsymbol{v}_D \times \boldsymbol{B}$ in the induction equation

$$\begin{aligned}\partial \boldsymbol{B}/\partial t &= \nabla \times (\boldsymbol{v}_i \times \boldsymbol{B} - \lambda \nabla \times \boldsymbol{B}) \\ &= \nabla \times (\boldsymbol{v}_n \times \boldsymbol{B} + \boldsymbol{v}_D \times \boldsymbol{B} - \lambda \nabla \times \boldsymbol{B}) , \end{aligned} \tag{1}$$

where λ is the ordinary magnetic diffusivity, and $\boldsymbol{v}_D \equiv \boldsymbol{v}_i - \boldsymbol{v}_n$ the drift velocity between ions and neutrals (e.g. Shu 1983; Zweibel 1987). The drift velocity is proportional to the Lorentz force with

$$\boldsymbol{v}_D \approx (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}/(4\pi \rho_i \nu_{in}) . \tag{2}$$

Here ρ_i is the mass density of ions and ν_{in} the ion-neutral collision frequency, which has to be short compared to the macroscopic timescales for the approximation (2) to be valid. Inserting equation (2) into equation (1) we obtain

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[\boldsymbol{v}_n \times \boldsymbol{B} + \frac{(\nabla \times \boldsymbol{B}) \cdot \boldsymbol{B}}{4\pi \rho_i \nu_{in}} \boldsymbol{B} - (\lambda + \lambda_{\rm AD}) \nabla \times \boldsymbol{B} \right] , \tag{3}$$

where $\lambda_{\rm AD} = \boldsymbol{B}^2/(4\pi \rho_i \nu_{in})$ is the ambipolar diffusion coefficient. Obviously, ambipolar diffusion introduces not only a variable magnetic diffusion, but the electric field also has a component in the direction of the magnetic field, which is reminiscent of the α-effect in mean-field dynamo theory (e.g., Krause & Rädler 1980; Zweibel 1988).

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There are two effects that can lead to the formation of sharp gradients: first, since $\boldsymbol{v}_D$ has a component proportional to the magnetic pressure gradient, $-\nabla(\boldsymbol{B}^2/8\pi)$, more and more flux will be pushed toward magnetic nulls, and second, $\lambda_{\rm AD}$ vanishes in magnetic nulls, which prevents the field from becoming smoothed out, thus making those points singular. Furthermore, while ordinary diffusion is able to change the field topology, ambipolar diffusion is not. This can be seen from the original equation (1), which shows that magnetic field lines are strictly attached to the ions.

We nondimensionalize by putting $4\pi\rho_i \nu_{in} = 1$, that is, we measure $\boldsymbol{B}$ in units of a reference value B_0, and time in units of the diffusion time $\tau = L^2/\lambda_{\rm AD}^{(0)}$, where L is the unit length, and $\lambda_{\rm AD}^{(0)} = B_0^2/(4\pi\rho_i \nu_{in})$ a reference value for the ambipolar diffusion coefficient.

In this *Letter* we give explicit examples of cases where ambipolar diffusion does lead to the development of sharp structures. We first study a one-dimensional example in which both the field direction and amplitude evolve steep gradients. We then consider the effect of ambipolar diffusion in two dimensions in the presence of a bulk velocity $\boldsymbol{v}_n$.

We note that although our deviation of a nonlinear diffusion coefficient is physically motivated by ion-neutral slip, similar diffusion terms could be invoked in simulations to stabilize the flow. As our results show (see, e.g., Fig. 5 and Plates L7 and L8), this procedure may have the unintended consequence of artificially steepening the gradients in some places.

2. SHARP STRUCTURES IN ONE DIMENSION

In the one-dimensional case, $\boldsymbol{B} = \hat{\boldsymbol{x}}B_x(z, t) + \hat{\boldsymbol{y}}B_y(z, t)$, it is convenient to write the equation in complex form using $\mathscr{B} \equiv B_x + iB_y$. For $\boldsymbol{v}_n = \lambda = 0$, assuming that the ionization fraction is very small the induction equation can be written in the form

$$\frac{\partial \mathscr{B}}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\mathscr{B}}{2}\frac{\partial |\mathscr{B}|^2}{\partial z}\right). \quad (4)$$

The neglect of the bulk velocity corresponds to the limit in which the magnetic field is weak and the medium is in overall pressure (or hydrostatic) equilibrium. This is not generally the case in the interstellar medium. However, in the example studied here, bulk flow down the magnetic pressure gradient only accentuates the tendencies found for ambipolar diffusion alone.

Apart from the trivial solution, $\mathscr{B}$ = constant, equation (4) permits steady solutions of the form $\mathscr{B} \propto z^{1/3}$. This solution has infinite current density at $z = 0$ ($\mathscr{J} = i\,\partial\mathscr{B}/\partial z \propto z^{-2/3}$). Furthermore, since it diverges at infinity, it can only be realized locally. In Figure 1 we show numerically how this solution is approached from an initial condition $\mathscr{B} \propto z$. In order to simulate an unbounded domain, we adopt an open boundary condition of the form $\partial\mathscr{B}/\partial z = \mathscr{B}/(3z)$, which is consistent with the solution anticipated in the computational domain. A nonuniform mesh is employed to improve resolution at $z = 0$. Away from $z = 0$ too fine a mesh would adversely affect the maximal time step, $\delta t_{\max} \sim \delta x^2/|\mathscr{B}|^2$. We have found similar steepening in the neighborhood of nulls if the initial amplitude of $\mathscr{B}$ is sinusoidal and periodic boundary conditions are used; this is consistent with unpublished integrations by W. Merryfield (1990, private communication).

We have carried out a linear stability analysis of the $z^{1/3}$ solution. We find that the only linear modes that are finite at $z = 0$ and either decay at infinity or satisfy homogeneous

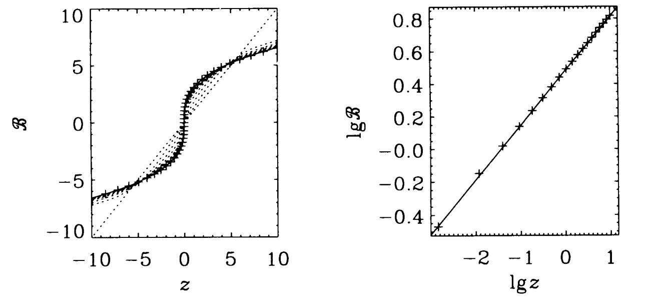

FIG. 1.—*Left*: Evolution of $\mathscr{B}$ (*dotted lines*) for the initial condition $\mathscr{B} \propto z$ (*dashed line*). The final solution (*solid line*) shows a $z^{1/3}$ behavior ($t = 1$). The time between the different lines is 0.1. *Right*: The final solution in a log-log plot.

boundary conditions on a finite domain are exponentially damped, so this solution is linearly stable.

Next, we consider an initial condition with nonuniform phase. It is easily shown from equation (4) that the amplitude of $\mathscr{B}$ evolves independently of the phase ϕ of $\mathscr{B}$, while ϕ is advected at the drift velocity v_D (Zweibel 1994). In Figure 2 we plot the evolution of the phase $\phi = \arctan(B_y/B_x)$ for the initial condition $\mathscr{B} \propto |z|^{1/3} \exp(ikz)$. This solution is in perfect agreement with the analytical solution $\phi(z, t) = \pm k(z^{4/3} + 4t/9)^{3/4}$ (Zweibel 1994; but note that the coefficient of t should read 4/9 instead of 9/4). A very similar behavior also occurs if the profile of the amplitude $|\mathscr{B}|$ was initially different, for example, proportional to $|z|$ as in the previous example. This is because the evolution toward $|\mathscr{B}| \sim |z|^{1/3}$ happens much faster than the evolution of the jump in the phase. This steepening does not occur unless $\mathscr{B}$ has a null.

These examples illustrate that it is in principle possible to generate sharp structures of the magnetic field due to ambipolar diffusion. However, the significance of these examples is limited, because they require the presence of magnetic nulls where all field components vanish simultaneously. In the absence of nulls in the initial condition, ambipolar diffusion alone is unable to produce any null points, but it rather acts as to make the field force free.

In the following we investigate an example in two dimensions where an imposed bulk velocity $\boldsymbol{v}_n$ is included. In this case it is possible to obtain (at least asymptotically) magnetic nulls.

3. AMBIPOLAR DIFFUSION IN TWO DIMENSIONS

The evolution of the magnetic field in two dimensions in the plane of the velocity $\boldsymbol{v}_n$ is governed by the equation (nondimensional)

$$DA/Dt = \boldsymbol{B}^2\nabla^2 A\,, \quad (5)$$

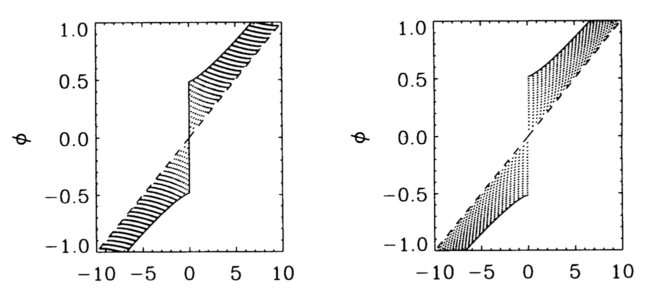

FIG. 2.—*Left*: Evolution of ϕ (*dotted lines*) for the initial condition $\phi = kz$ with $k = 0.1$ (*dashed line*). The solid line is for $t = 10$. *Right*: The analytical solution for comparison.

where $D/Dt = \partial_t + \boldsymbol{v}_n \cdot \nabla$ is the total derivative, $A(x, y, t)$ the flux function, and $\boldsymbol{B} = \nabla \times (A\hat{z})$. If equation (5) is solved on a finite domain with periodic or perfectly conducting and stress-free boundary conditions, or in an infinite domain with B approaching zero at infinity, then it can be shown that the total magnetic energy decays with time according to the equation

$$\frac{d}{dt}\int \frac{\boldsymbol{B}^2}{2}\,dx\,dy = -\int \boldsymbol{J}^2\boldsymbol{B}^2\,dx\,dy\,. \qquad (6)$$

Although the total magnetic energy decays under these conditions, the current density, $\boldsymbol{J} = -\hat{z}\nabla^2 A$, can increase and singularities similar to those found in the $1-D$ case can arise.

Here we study the effect of ambipolar diffusion on magnetic field lines entrained by a rotating eddy in nonuniform, circular motion. In the absence of any sort of diffusion A is advected with the flow, and equation (5) has the solution $A(r, \theta, t) = A_0[r, \theta - \Omega(r)t]$, where A_0 is the initial value of A and $\Omega \equiv v/r$. If the magnetic field lines are initially straight lines, the field winds up with time. If resistive diffusion is added, the windup is prevented because diffusion becomes progressively more important as the field develops structure on fine scales. In order to highlight the differences between resistive and ambipolar diffusion, we model the latter using a time-dependent diffusion coefficient $\lambda(t) = \langle \boldsymbol{B}^2 \rangle$. This choice is motivated by the observation that in the absence of a bulk velocity the decay of $\langle \boldsymbol{B}^2 \rangle$ is then quantitatively similar in the two cases.

For the velocity we take a differentially rotating spiral with $\boldsymbol{v}_n = \nabla \times (\psi\hat{z})$ and $\psi = (\cos \pi r/2)^4$ for $r^2 = x^2 + y^2 < 1$ and $\psi = 0$ otherwise. The initial condition is $A = B_0 y$ with $B_0 = 0.1$. We solve for the deviations of A from the initial state using a uniform, periodic, staggered, 128^2 or 512^2 mesh of sixth order with a third order Hyman time step (Nordlund 1991). In Figure 3 (Plate L7) we compare the evolution of magnetic field lines (contours of A) and the current density (color coded) with the corresponding cases of resistive magnetic diffusion, and without any diffusion (ideal case). In Figure 4 (Plate L8) we show the central parts of the magnetic field and current density under the influence of ambipolar diffusion and with ideal evolution. In Figures 3, 4, and 6, time is measured in units of the rotation period at $r = 0$, $T_{\rm rot} = 2\pi/\Omega(r = 0)$. The effects of resistive diffusion are very clear: the field is not wound up. The implied values of the resistivity in this case are of course unrealistically large for the interstellar medium.

In the case of ambipolar diffusion the magnetic field is advected by the flow in a similar manner as in the ideal case without diffusion. However, there is a marked difference between the two in that the current density only occupies very narrow regions in space and forms current filaments, between which the magnetic field is essentially current-free. (In fact, the tendency for ambipolar diffusion to produce current-free, or possibly force-free in three dimensions, states suggests its use as a computational device to calculate magnetostatic equilibria.) In the ideal case, on the other hand, no such pronounced filaments occur, and the field is wound up more uniformly. This is clearly shown in Figure 5, which shows the current on a slice through the center. Notice that although the peaks and valleys correspond one to one in these two cases, ambipolar diffusion produces spikes in the current. Nevertheless, these two cases have in common invariant magnetic field topology, which is in clear contrast to the case of ordinary diffusion.

In a first attempt to investigate the nature of magnetic reconnection in a wound-up magnetic field, we carried out a tearing mode stability analysis of the current profiles shown in Figure 5 (White 1983). The tearing mode is driven by short length scales for the variation of the current, and the wavelength of the unstable modes must generally exceed the width of the magnetic shear layer, so we were not surprised to find that the tearing mode growth rates in the ambipolar case appear to be enhanced by more than two orders of magnitude over the ideal case once the field has wound up ($t > T_{\rm rot}$). However, our models do not completely satisfy the assumptions of tearing mode theory. The theory assumes one-dimensional slabs of field in static equilibrium which are completely symmetrical across the null line. In our models the field is evolving dynamically, the lines are curved, and especially in the ideal case the current profiles show significant asymmetry. Therefore we prefer not to make quantitative comparisons of the reconnection rates, but to let the sharp current spikes in the left panel of Figure 5 speak for themselves.

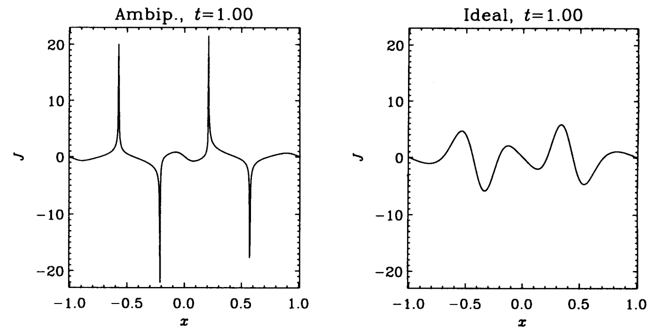

FIG. 5.—Comparison of the profiles of J vs. x at $y = 0$ for $t = T_{\rm rot}$ for the ambipolar case and the ideal case.

Finally we compare some statistical properties of the three cases (Fig. 6). While the magnetic energy keeps increasing in the ideal case, it behaves remarkably similar in the cases of ambipolar and ordinary diffusion. It turns out that to a good approximation

$$\frac{d}{dt}\left\langle \frac{\boldsymbol{B}^2}{2} \right\rangle = -\langle \boldsymbol{B}^2\boldsymbol{J}^2 \rangle \approx -\langle \boldsymbol{B}^2 \rangle \langle \boldsymbol{J}^2 \rangle\,, \qquad (7)$$

even though the evolution of the actually fields is rather different for ambipolar and ordinary diffusion. The results in Figure 6 were obtained for 128^2 points; a 512^2 grid shows remarkably similar behavior.

In physical units the ambipolar diffusion timescales can be estimated as follows. Using

$$\lambda_{\rm AD} = 2.4 \times 10^{19}\,\frac{B_\mu^2}{A_r n_i n_n}\,{\rm cm^2\,s^{-1}} \qquad (8)$$

(e.g., Zweibel 1987), where B_μ is the magnetic field in μG, n_i and n_n are the ion and neutral particle densities in ${\rm cm^{-3}}$, and A_r is

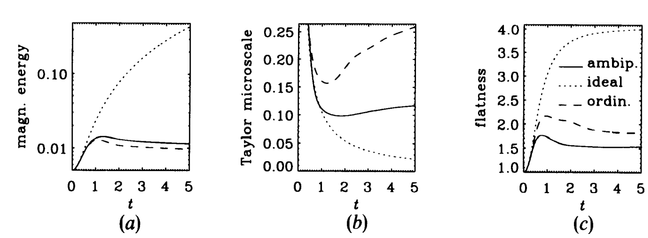

FIG. 6.—Evolution of rms values of (a) the magnetic energy, (b) the magnetic Taylor microscale, $\lambda_M = B_{\rm rms}/J_{\rm rms}$, and (c) the flatness, $\langle \boldsymbol{B}^4 \rangle / \langle \boldsymbol{B}^2 \rangle^2$, of the magnetic field.

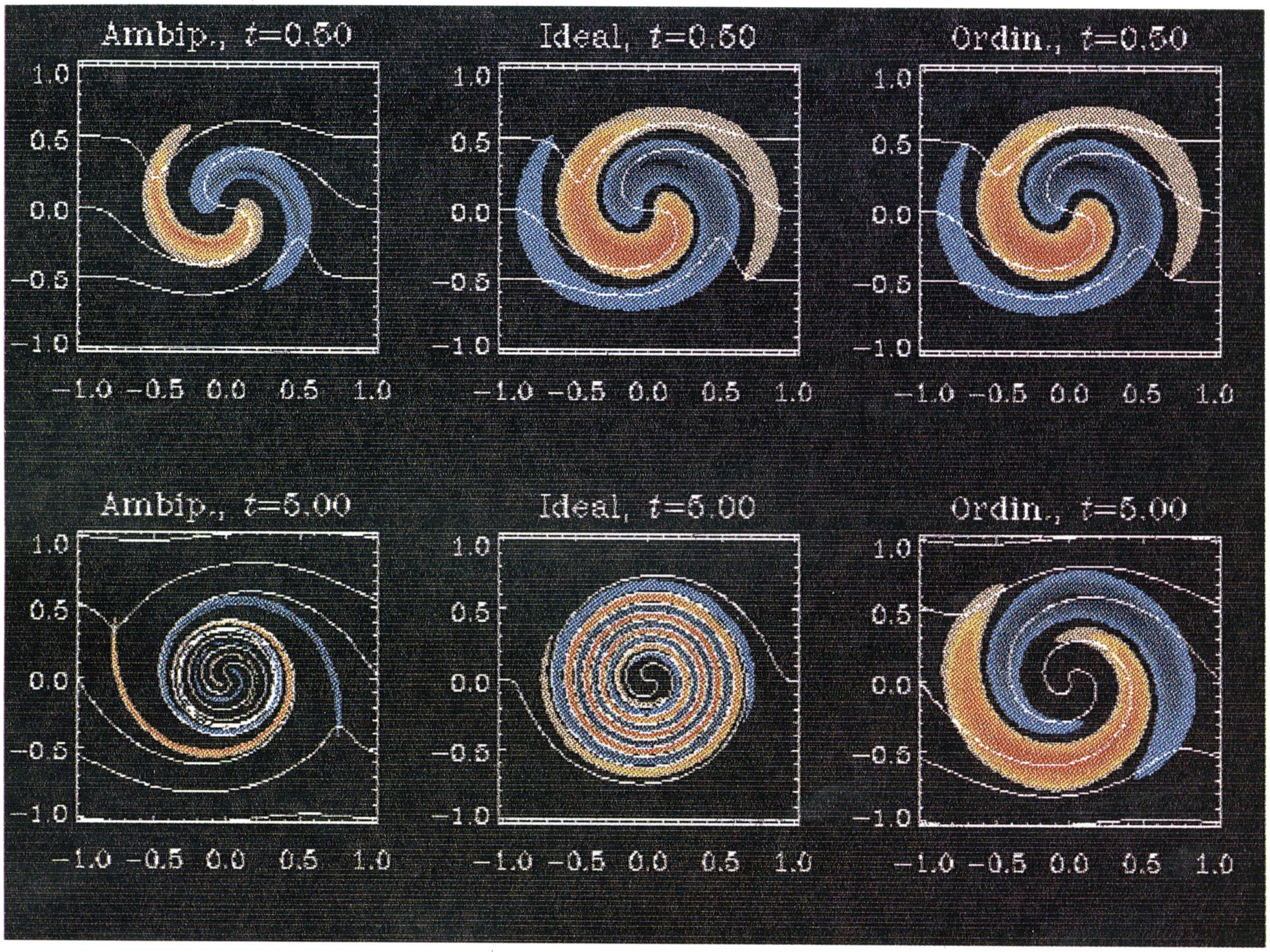

FIG. 3.—Comparison of the effects of ambipolar and ordinary diffusion with the ideal case at two different times (in units of T_{rot}) for a spiral-like velocity field. Magnetic field lines are plotted in white, and the current density is color coded (red for positive values and blue for negative). The resolution is 128^2 points.

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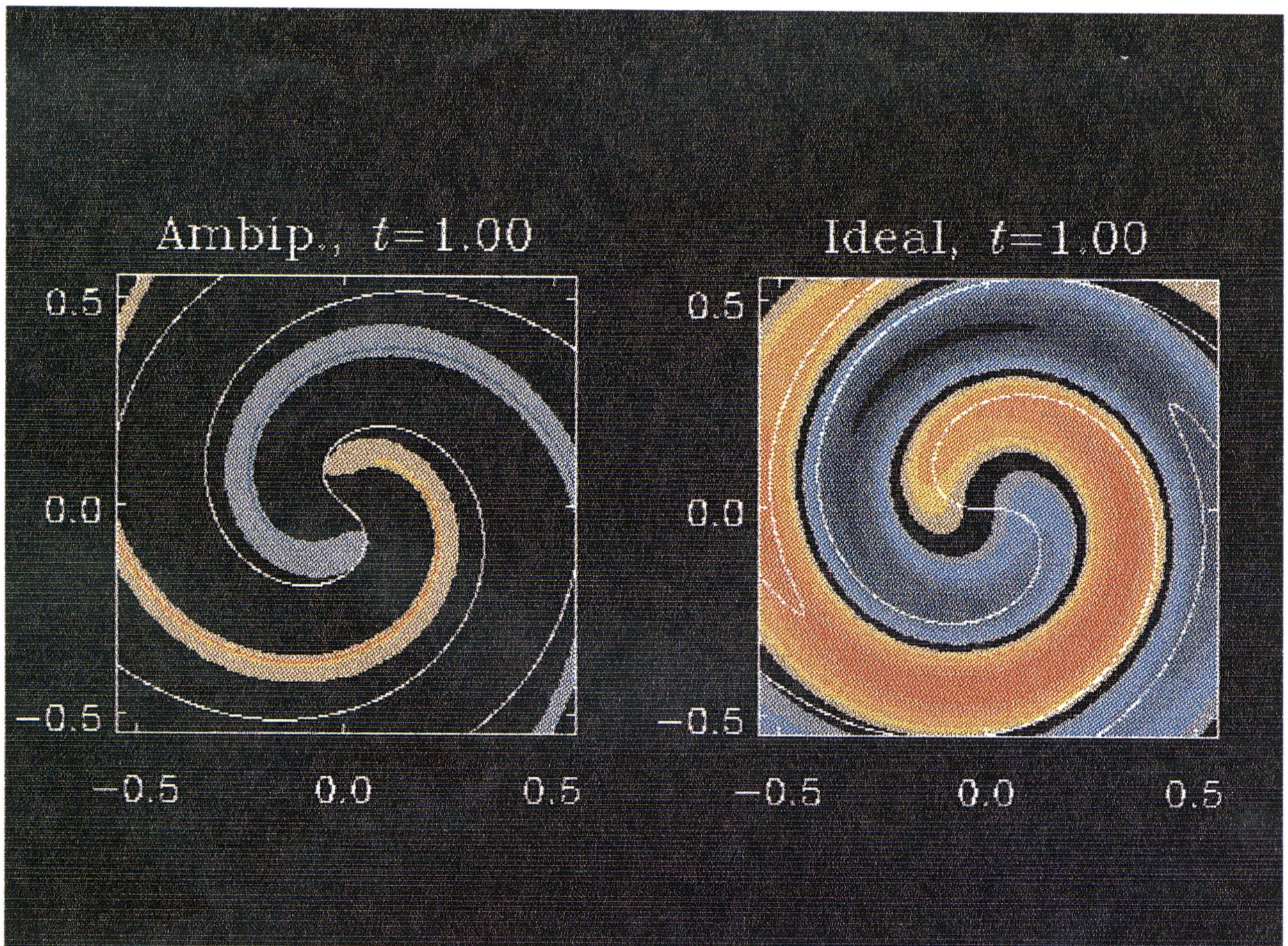

FIG. 4.—Comparison of ambipolar and ideal cases for the central parts of the domain at $t = T_{rot}$. Note that in all cases the color scale is normalized with respect to the maximum value of J. The resolution is 512^2 points.

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the reduced particle mass, we have $\lambda_{AD} \approx 10^{23}$ cm^2 s^{-1} for molecular clouds ($B_\mu = 30$, $n_n = 10^3$, $n_i = 3 \times 10^{-4}$). We find a similar value for diffuse gas ($B_\mu = 3$, $n_n = 1$, $n_i = 10^{-3}$). Taking as a typical length scale $L = 1$ pc, $[t] = L^2/\lambda_{AD} \approx 4 \times 10^6$ yr. The velocity unit is $[u] = \lambda_{AD}/L \approx 1.4$ km s^{-1}. In our numerical experiments the nondimensional velocity was of order unity, and thus the dimensional velocities in that case would be a typical value for clouds.

4. LIMITS OF VALIDITY AND APPLICATIONS

We have shown that magnetic nulls are likely to be steepened by ambipolar diffusion, and that a magnetic field stretched by shear flows will also display current filamentation even without exact null points. We suggest that the large current densities associated with these steep gradients can lead to rapid magnetic field line reconnection, possibly with ions and neutrals decoupled from each other (Zweibel 1989). As we noted in the introduction, reconnection is an essential ingredient in hydromagnetic dynamos. Reconnection is also expected to occur during the star formation process, possibly in magnetic neutral sheets formed during gravitational collapse (Mestel 1966; Galli & Shu 1993). It has also been suggested that reconnection occurs in clumpy molecular clouds as the magnetic field lines are dragged about by the turbulent gas (Clifford & Elmegreen 1983). The difficulty of reconciling phenomenologically motivated arguments for reconnection with the very large magnetic Reynolds numbers encountered in astrophysical systems has been a long-standing problem. Although true magnetic null points may occur relatively rarely, velocity shear is probably more easily come by. Thus, we expect significant enhancement of the magnetic reconnection rate in turbulent, weakly ionized media.

As happens in general when ideal equations predict the formation of a singularity, we expect other physical effects to come into play. Equation (1) must be modified to account for the dependence of ν_{in} on the drift speed when v_D exceeds about 20 km s^{-1} (Draine, Roberge, & Dalgarno 1983). Note that $v_D \propto z^{-1/3}$ for $|B| \propto z^{1/3}$, so equation (1) will certainly break down near the null. However, if one sets $\nu_{in} \propto v_D$ (Draine et al. 1983), then one might expect the profile near the null to evolve to a $z^{1/4}$ dependence, which is even more singular. However, inertial effects also become important for large drift speeds, and this, as well as the effect of resistivity, will prevent singularity formation. Ion pressure could also slow down the collapse, but for a weakly ionized medium this should be a small effect.

We thank Steve Arendt for suggesting improvements to the manuscript. E. G. Z. is happy to acknowledge support by NSF and NASA grants, a University of Colorado Faculty Fellowship, and the hospitality of HAO, where part of this work was completed.

REFERENCES

Beck, R. 1993, in Galactic Dynamos: A Challenge for Observers (The Cosmic Dynamo), ed. F. Krause et al. (Dordrecht: Kluwer), 283
Cattaneo, F., & Vainshtein, S. I. 1991, ApJ, 376, L21
Clifford, P., & Elmegreen, B. G., 1983, MNRAS, 202, 629
Draine, B. T., Roberge, W. G., & Dalgarno, A. 1983, ApJ, 264, 485
Galli, D., & Shu, F. H. 1993, ApJ, 417, 220
Krause, F., & Rädler, K.-H. 1980, Mean-Field Magnetohydrodynamics and Dynamo Theory (Berlin: Akademie-Verlag)
Krause, F., & Rüdiger, G. 1975, Sol. Phys., 42, 107
Kulsrud, R. M., & Anderson, S. W. 1992, ApJ, 396, 606
Low, B. C. 1982, Phys. Fluids, 25, 402
Mestel, L. 1966, MNRAS, 133, 265
Nordlund, Å. 1991, Computer Physics, unpublished lecture notes, Univ. Copenhagen
Parker, E. N. 1979, Cosmical Magnetic Fields (Oxford: Clarendon Press)
Parker, E. N. 1992, ApJ, 401, 137
Proctor, M. R. E., & Zweibel, E. G. 1992, Geophys. Astrophys. Fluid Dyn., 64, 145
Roberts, P. H., & Soward, A. M. 1975, Astron. Nach. 296, 49
Ruzmaikin, A. A., Shukurov, A. M., & Sokoloff, D. D. 1988, The magnetic Fields of Galaxies (Dordrecht: Kluwer)
Shu, F. H. 1983, ApJ, 273, 202
White, R. B. 1983, in Handbook of Plasma Physics, ed. M. N. Rosenbluth & R. Z. Sagdeev (Amsterdam: North-Holland), 611
Zweibel, E. G. 1987, in Interstellar Processes, ed. D. J. Hollenbach & H. A. Thronson, Jr. (Dordrecht: Reidel), 195
———. 1988, ApJ, 329, 384
———. 1989, ApJ, 340, 550
———. 1994, in Cosmical Magnetic Fields, ed. D. Lynden-Bell (Cambridge: Cambridge Univ. Press), in press