New tests for a singularity of ideal MHD

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Analysis using new calculations with 3 times the resolution of the earlier linked magnetic flux tubes confirms the transition from singular to saturated growth rate reported by Grauer and Marliani \cite{2} for the incompressible cases is confirmed. However, all of the secondary tests point to a transition back to stronger growth rate at a different location at late times. Similar problems in ideal hydrodynamics are discussed, pointing out that initial negative results eventually led to better initial conditions that did show evidence for a singularity of Euler. Whether singular or near-singular growth in ideal MHD is eventually shown, this study could have bearing on fast magnetic reconnection, high energy particle production and coronal heating.

The issue currently leading to conflicting conclusions about ideal 3D, incompressible MHD is similar \cite{4,5} to what led to conflicting results on whether there is a singularity of the 3D incompressible Euler. With numerical simulations, it was first concluded that uniform mesh calculations with symmetric initial conditions such as 3D Taylor-Green were not yet singular \cite{2}. Next, a preliminary spectral calculation \cite{1} found weak evidence in favor of a singularity in a series of Navier-Stokes simulations at increasing Reynolds numbers, but larger adaptive mesh or refined mesh calculations did not support this result \cite{5,6}. Eventually, numerical evidence in favor of a singularity of Euler was obtained using several independent tests applied to highly resolved, refined mesh calculations of the evolution of two anti-parallel vortex tubes \cite{7}. To date, these calculations have met every analytic test for whether there could be a singularity of Euler.

Several other calculations have also claimed numerical evidence for a singularity of Euler \cite{4,5,6,7}. While in all of these cases the evidence is plausible, with the perturbed cylindrical shear flow \cite{1} using the BKM $\|\omega\|_\infty$ test \cite{2}, for none has the entire battery of tests used for the anti-parallel case been applied. We have recently repeated one of the orthogonal cases \cite{5} and have applied the BKM test successfully. In all cases using the BKM test, $\|\omega\|_\infty \approx A/(T_c - t)$ with $A \approx 19$.

To be able to make a convincing case for the existence of a singularity in higher dimensional partial differential equations, great care must be taken with initial conditions, demonstrating numerical convergence, and comparisons to all known analytic or empirical tests. On the other hand, if no singularity is suspected, some quantity that clearly saturates should be demonstrated, such as the strain causing vorticity growth \cite{4}. It is an even more delicate matter to claim that someone else’s calculations or conclusions are incorrect. If it is a matter of suspecting there is inadequate resolution, one must attempt to reproduce the suspicious calculations as nearly as possible and show where inadequate resolution begins to corrupt the calculations and how improved resolution changes the results.

An example of how a detailed search for numerical errors should be conducted can be found in the extensive conference proceeding \cite{12} that appeared prior to the publication of the major results supporting the existence of a singularity of Euler for anti-parallel vortex tubes \cite{7}. The primary difference with earlier work was in the initial conditions. It was found that compact profiles \cite{13} were an improvement, but only if used in conjunction with a high wavenumber filter. Otherwise, the initial unfiltered energy spectrum of the bent anti-parallel vortex tubes went as $k^{-2}$. Oscillations in the spectrum at high wavenumber in unfiltered initial conditions for linked magnetic flux tubes are shown in Figure 1, showing that the initial MHD spectrum is steep enough that eventually these oscillations are not important.

The purpose of this letter is to address the claim that a new adaptive mesh refinement (AMR) calculation by Grauer and Marliani \cite{2} supersedes our uniform mesh calculations \cite{1} and that eventually there is a transition to exponential growth. Note that this claim was made without any evidence for whether their numerical method was converged. In all of our earlier calculations, once the calculations become underresolved, we also saw transitions to exponential growth.

Not knowing exactly the initial condition used by the new AMR calculations \cite{2}, where and how much grid re-
finement was used, and the short notice we have been given to reply has proven a challenge. Fortunately, we were in the process of new 648\(^3\) calculations in a smaller domain of 4.3\(^3\), yielding effectively 3 times the local resolution of our earlier work [3] in a (2\(\pi\))\(^3\) domain on a 384\(^3\) mesh. The case with an initial flux tube diameter of \(d = 0.65\), so that the tubes slightly overlap, appears to be closer to their initial condition and so will be the focus of this letter. The importance of our other initial condition, with \(d = 0.5\), and no initial overlap of the tubes, is that it is less influenced by an initial current sheet that forms near the origin and is claimed to be the source of the saturation of the nonlinear terms. This was used for the compressible calculations.

Using semi-log coordinates, Figure 2 plots the growth of \(\|J\|_\infty\) and \(\|\omega\|_\infty\) for our new high resolution incompressible calculation and Figure 3 plots \(\|J\|_\infty\) for a new compressible calculation. By taking the last time all relevant quantities on the 384\(^3\) and 648\(^3\) grids were converged, \(\|J\|\) being the worst, then by assuming that the smallest scales are decreasing linearly towards a possible singular time, an estimate of the last time the 648\(^3\) calculation was valid was made. To test exponential versus inverse linear growth, fits were taken between \(T = 1.72\) and 1.87, then extrapolated to large \(T\). The large figure shows that either an exponential or a singular \(1/(T_c - t)\) form could fit the data, while the inset shows that taking an estimated singular time of \(T_c = 2.15\) and multiplying by \((T_c - T)\) that at least \(\|J\|_\infty\) and \(\|\omega\|_\infty\) have consistent singular behavior over this time span. The strong growth of \(P_{ij} = \int dV (\omega_i e_{ij}) J_i - \omega_i d_{ij} J_j - 2\varepsilon_{ijk} J_i d_{jk} e_{ik}\), which is the production of \(\int dV (\omega^2 + J^2)\), is discussed below. The 384\(^3\) curve for \(1/\|J\|\) demonstrates that lack of resolution tends to exaggerate exponential growth. For the compressible calculations it can be seen that there also is an exponential regime that changes into a regime with \(1/\|J\|_\infty \sim (T_c - t)\).

FIG. 2. Replot of \(\|J\|_\infty\), \(\|\omega\|_\infty\), and \(P_{ij}\) for the new incompressible calculations on 4.3\(^3\) domain with initial condition \(d = 0.65\) in semi-log coordinates. All plots are from the 648\(^3\) calculation except one 384\(^3\) plot of \(1/\|J\|\). Exponential and inverse linear fits are shown for \(t = 1.75\) to 1.98. Each works equally well for \(\|J\|_\infty\), inverse linear is better for \(\|\omega\|_\infty\), and exponential is better for \(P_{ij}\). Multiplying \(\|J\|_\infty\), \(\|\omega\|_\infty\), and \(P_{ij}\) by \((T_c - t)\) in the inset emphasizes that \(\|J\|_\infty\) and \(\|\omega\|_\infty\) might be showing consistent singular behavior. The large growth of \(P_{ij}\) is discussed.

Using the new incompressible calculations and applying the entire battery of tests, based upon Figure 2 we would agree that for the incompressible case there is a transition as reported [4] and signs of saturation at this stage are shown below. Whether the transition is to exponential for all times as claimed [2], or whether there is a still later transition to different singular behavior, will be the focus of this letter. We will look more closely at the structure of the current sheet we all agree exists [5] for signs of saturation.

The case against a singularity in early calculations of Euler [10,14,15] was the appearance of vortex sheets, and through analogies with the current in 2D ideal MHD, a suggestion that this leads to a depletion of nonlinearity. The fluid flow most relevant to the linked flux rings is 3D Taylor-Green, due to the initial symmetries [6]. For both TG and linked flux tubes, two sets of anti-parallel vortex pairs form that are skewed with respect to each other and are colliding. In TG, just after the anti-parallel vortex tubes form there is a period of slightly singular development. This is suppressed once the pairs collide with each other, and then vortex sheets dominate for a period. The vortex sheets are very thin, but go across the domain, so fine localized resolution might not be an

FIG. 3. Semi-logarithmic plot of \(\|J\|_\infty\) for a compressible 240\(^3\) calculation in a domain of size 4 (dotted line: filtered, and solid line: unfiltered initial conditions) together with fits to exponential growth and blow-up behavior, respectively. The latter are better fits at later times.
advantage at this stage. At late phases in TG, the ends of
the colliding pairs begin to interact with each other,
so that at 4 corners locally orthogonal vortices begin to
form. Due to resolution limitations, an Euler calculation
of Taylor-Green has not been continued long enough to
determine whether, during this phase, singular behavior
might develop. We would draw a similar conclusion for
all of MHD cases studied to date [2, 16, 17], that there
might not be enough local resolution to draw any final
conclusions even if AMR is applied.

While Taylor-Green has not been continued far enough
to rule out singularities, the final arrangement of vortex
structures led first to studies of interacting orthogonal
vortices [8], and then anti-parallel vortices (see references
in [8]). Both of these initial conditions now appear to de-
velop singular behavior. An important piece of evidence
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The initial singular growth of \( |J|_{\infty} \) and \( |\omega|_{\infty} \) for
the linked flux rings, then the transition to a saturated
growth rate, might be due to the same skewed, anti-
parallel vortex pair interaction as in Taylor-Green. Even
if this is all that is happening, the strong initial vortic-
ity production and shorter dynamical timescale (order of
a few Alfvén times) than earlier magnetic reconnection
simulations with anti-parallel flux tubes [17] is a signif-
icant success of these simulations. It might be that the
vortices that have been generated are strong enough to
develop their own Euler singularity. However, the in-
teresting physics is how the magnetic field and current
interact with the vorticity. Do they suppress the ten-
dency of the vorticity to become singular, or augment
that tendency?

One sign for saturation of the linked flux ring inter-

action would be if the strongest current remains at the
origin in this sheet. Figure 4 plots the positions of \( |J|_{\infty} \)
and \( |\omega|_{\infty} \) from the origin as a function of time. Dur-
ing the period where exponential growth is claimed [2],
\( |J|_{\infty} \) is at the origin, which would support the claims of
saturation. However, this situation does not persist.

By analogy to the movement of the \( L^\infty \) norms of the
components of the stress tensor \( u_{i,j} \) in Euler, we expect
that the positions of \( |J|_{\infty} \) and \( |\omega|_{\infty} \) should approach
each other and an extrapolated singular point in ideal
MHD. Figure 5 supports the prediction that the positions
of \( |J|_{\infty} \) and \( |\omega|_{\infty} \) should approach each other but so far
not in a convincingly linear fashion. This is addressed
next. We have similar trends for the positions of \( |J|_{\infty} \)
and \( |\omega|_{\infty} \) in the compressible calculations.

![Current sheet and vectors](image)

**FIG. 4.** Positions of \( |J|_{\infty} \) and \( |\omega|_{\infty} \) for \( d = 0.65 \) in a 4.3³
domain.

![Graph](image)

**FIG. 5.** For \( t = 1.97 \) on the inner 162³ grid points, the cur-
rent sheet is shown with arrows of \( \vec{J} \) overlaid in dark. The
current through the \((x/y = z)\) plane containing \( |J|_{\infty} \) is in
lower right. Contours of \( |J| \) are shown to emphasize where
\( |J|_{\infty} \) is located. Dark lines are \( \vec{B} \) and light lines are \( \vec{\omega} \)
that originated in the vicinity of \( |J|_{\infty} \). The vortex lines are
predominantly those in the double vortex rings that were ori-
ginally generated by the Lorenz force, then became responsible
for spreading out the current sheet. Where the \( \vec{B} \) lines cross
in the upper left and lower right corners are around the lo-
cations of \( |J|_{\infty} \), which due to symmetries are different views
of the same structure. Near \( |J|_{\infty} \), \( \vec{B} \) nearly overlies and is
parallel to \( \vec{\omega} \) and both \( \vec{B} \) and \( \vec{\omega} \) are nearly orthogonal to
their partners across the current sheet, where \( \vec{B} \) and \( \vec{\omega} \) are
anti-parallel. Taken from the \( d = 0.65 \) calculation in a 4.3³
domain on a 648³ mesh.
As noted, for Euler all available calculations find \( \| \omega \|_{\infty} \approx A/(T_e - t) \) with \( A \approx 19 \). \( A \) represents how much smaller the strain along \( \| \omega \|_{\infty} \) is than \( \| J \|_{\infty} \). Here, \( A \approx 4 \), indicating stronger growth in \( \| \omega \|_{\infty} \) for ideal MHD than Euler. Another Euler result was that the asymptotic energy spectrum as the possible singularity was approached was \( k^{-3} \), whereas purely sheet-like structures in vorticity should yield \( k^{-4} \) spectrum. \( k^{-3} \) indicates a more complicated 3D structure than sheets. In Figure 4 the late time spectra are again \( k^{-3} \).

The next the initial condition we will investigate will be magnetic flux and vortex tubes that nearly overlay each other and are orthogonal to their partners. Our new calculations of orthogonal vortex tubes for Euler show that they start becoming singular as filaments are pulled off of the original tubes and these filaments become anti-parallel, suggesting that the fundamental singular interaction in Euler is between anti-parallel vortices. Whether the next step for ideal MHD is to become anti-parallel or something else can only be determined by new calculations. AMR might be useful, but great care must be taken with the placement of the inner domains and a large mesh will still be necessary. The complicated structures in the domain in Figure 3 are not fully contained in this innermost 16\(^2\) mesh points and the innermost domain should go out to the order of 300\(^3\) points. There are examples of how to use AMR when there are strong shears on the boundaries of sharp structures [18]. This uncertainty of where to place the mesh is why we believe in using uniform mesh calculations as an unbiased first look at the problem.

These final results are hardly robust and their usefulness is primarily to suggest a new more localized initial condition and to show that none of the calculations to date is the full story. For \( J \) and \( \omega \) to show singular behavior as long as they have been surprising. Recall that for Euler, velocity, vorticity and strain are all manifestations of the same vector field, but for ideal MHD there are two independent vector fields even though the only analytic result in 3D is a condition on the combination, \( \int dV \left( \| \omega \|_{\infty}(t) + \| J \|_{\infty}(t) \right) dt \to \infty \) [19]. Eventually, one piece of evidence for singular growth must be a demonstration of strong coupling between the current and vorticity so that they are acting as one vector field. It could be that our strong growth is due to the strongly helical initial conditions and there are no singularities. This would still be physically interesting since helical conditions could be set up by footpoint motion in the corona.

Could the magnetic and electric fields blow up too? There are signs this might be developing around the final position of \( \| J \|_{\infty} \), in which case there might exist a mechanism for the direct acceleration of high energy particles. This has been considered on larger scale [22], but to our knowledge a mechanism for small-scale production of super-Dreicer electric fields has not been proposed before. A singular rise in electric fields could explain the sharp rise times in X-ray production in solar coronal measurements [21], which could be a consequence of particle acceleration coming from reconnection. This would also have implications for the heating of the solar corona by nanoflares [22] and the production of cosmic rays.

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