Helicity in Hydro and MHD Reconnection

Axel Brandenburg[1,2] and Robert M. Kerr[3,4]

[1] Department of Mathematics, University of Newcastle upon Tyne, NE1 7RU, UK
[2] NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
[3] NCAR, Boulder, CO 80307-3000, USA
[4] Department of Atmospheric Sciences, University of Arizona, Tucson, AZ 85721-0081, USA

Abstract. Helicity, a measure of the linkage of flux lines, has subtle and largely unknown effects upon dynamics. Both magnetic and hydrodynamic helicity are conserved for ideal systems and could suppress nonlinear dynamics. What actually happens is not clear because in a fully three-dimensional system there are additional channels whereby intense, small-scale dynamics can occur. This contribution shows one magnetic and one hydrodynamic case where for each the presence of helicity does not suppress small-scale intense dynamics of the type that might lead to reconnection.

1 Introduction

The term reconnection is used in both the MHD and fluids communities to describe topological changes in magnetic or vorticity fields due to resistivity or viscosity and could not occur in ideal cases where these dissipative terms are zero. In the strictly ideal limit the connectivity of the field lines would not change, but this is a singular limit and even the smallest amount of resistivity or viscosity allows the connectivity to change, albeit on small length scales. In the presence of finite dissipative terms, large amounts of energy can be converted into heat (or other forms of energy if one goes beyond the hydrodynamic approximations). MHD reconnection plays a role in understanding why the solar corona (i.e. the tenuous layers above the solar surface) are heated to $\sim 10^6$ K, even though at the surface of the sun the temperature is only ~ 6000 K. The other aspect is that the dissipative terms allow the field line connectivity to change. There are strong indications from observations of the solar corona in X-rays that field loops originally tied to the solar surface all of a sudden break loose and transport large amounts of flux into outer space. This raises the issue of how fast can reconnection occur. This is perhaps the single most challenging aspect of the problem.

Early work on MHD reconnection was concerned with steady state configurations, allowing a constant flux of material to pass through an X-point type configuration in two-dimensional field line configurations. However, because the reconnection site becomes very thin as the magnetic resistivity decreases, the amount of flux processed through the reconnection site decreases like the square root of the resistivity and by this mechanism finite reconnection in a dynamical timescale is not feasible for typical astrophysical values of resistivity. Other more

complicated initial conditions can lead to slow shocks that increase the reconnection rate, as discussed in a recent textbook [1]. Is not clear, however, whether the various boundary conditions studied so far represent anything physical in the corona and whether the results could explain the nanosecond timescales over which hard X-ray output associated with reconnection is seen to rise.

2   Dissipation of Energy and Helicity

Magnetic reconnection has two distinct aspects. One is the speed at which magnetic energy can be converted into heat and the other is the speed at which the magnetic topology can change. The two need not be the same. The perhaps worst possible type of topology to change is one that invokes mutual linkage of flux tubes, which can be described by the magnetic helicity H defined as

$$H = \int \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V, \tag{1}$$

where $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{A}$ is the vector potential such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Obviously, $\boldsymbol{A}$ is not uniquely defined, because adding an arbitrary gradient field to $\boldsymbol{A}$ would not change $\boldsymbol{B}$. However, the value of H is unaffected by this if the integral is taken over a domain where the normal component of the field vanishes on the boundaries. In that case

$$\int (\boldsymbol{A} + \operatorname{grad}\varphi) \cdot \boldsymbol{B} \, \mathrm{d}V = \int \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V + \int \varphi \boldsymbol{\nabla} \cdot \boldsymbol{B} \, \mathrm{d}V = H, \tag{2}$$

because the magnetic field is always solenoidal, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Another conserved quantity is the cross helicity, $H_\mathrm{c} = \int \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V$, which describes the linkage between flux tubes and vortex tubes. In the absence of magnetic fields the hydrodynamic helicity, $H_\mathrm{h} = \int \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V$, is conserved by the inviscid Euler equations, and it describes the linkage of vortex tubes with themselves.

The standard example that highlights the connection between magnetic helicity and topology is an interlocked pair of flux rings (see, e.g., the first panel of Fig. 1), for which the magnetic helicity is given by twice the product of the two magnetic fluxes of each of the two flux rings. However, helicity is also associated with two orthogonal flux tubes as shown in Fig. 5.

The dramatic difference between the dissipation of magnetic energy and magnetic helicity can best be seen by contrasting the equations of the conservation of magnetic energy and magnetic helicity,

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{B} \cdot \boldsymbol{B} \rangle = -\langle \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) \rangle - \eta \langle \boldsymbol{J} \cdot \boldsymbol{J} \rangle, \tag{3}$$

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = -\langle \boldsymbol{u} \cdot (\boldsymbol{B} \times \boldsymbol{B}) \rangle - \eta \langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle, \tag{4}$$

where angular brackets denote volume averages, and surface terms are assumed to vanish.
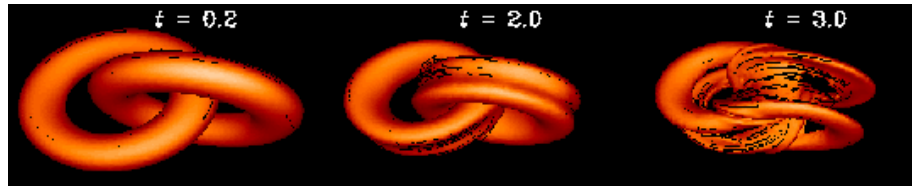
Fig. 1. Resistive evolution of an initially interlocked pair of flux rings. Isosurfaces of the magnetic field are shown at different times.

The important point to note here is that the magnetic energy can maintain a steady state where Joule dissipation, $\eta\langle J^2\rangle$, can be finite and large if work is done against the Lorentz force, i.e. if $-\langle u \cdot (J \times B)\rangle > 0$. At the same time, however, there is no such term in the magnetic helicity equation, so in that case a steady state is only possible if the current helicity, $\langle J \cdot B\rangle$, vanishes.

In the absence of any forcing, f, the hydrodynamic helicity is conserved in a similar manner, but there are two important differences. First, because $\int u \cdot \omega\, dV$ contains one more derivative than $\int u^2\, dV$, it dissipates faster than the energy if there is dissipation. Second, if there is forcing, the hydrodynamic helicity is no longer conserved. This difference to the hydromagnetic case can best be seen by contrasting eqs. (3) and (4) with the corresponding equations in hydrodynamics,

$$\frac{1}{2}\frac{d}{dt}\langle u \cdot u\rangle = \langle u \cdot f\rangle - \nu\langle \omega \cdot \omega\rangle, \tag{5}$$

$$\frac{1}{2}\frac{d}{dt}\langle u \cdot \omega\rangle = \langle \omega \cdot f\rangle - \nu\langle \theta \cdot \omega\rangle, \tag{6}$$

where $\theta = \nabla \times \omega$ is the curl of the vorticity, and surface terms are again assumed to vanish. Thus, unlike the magnetic counterpart, kinetic helicity conservation is only possible in the special case where the forcing is perpendicular to the vorticity and the flow is inviscid. With dissipation and the absence of forcing both kinetic energy and kinetic helicity are decaying, but kinetic helicity contains an extra derivative more than the kinetic energy and so decays faster than energy and does not pose a hard constraint.

3   Interlocked Flux Rings

In Fig. 1 we show an example of an initial flux tube configuration where, in our case, each tube has the flux $\Phi = \int B \cdot dS = 0.7B_0 d^2$, where B_0 is the maximum field strength in the core of each tube and d is its radius. The magnetic helicity is measured to be $H = \int A \cdot B\, dV = 0.98B_0^2 d^4$, in perfect agreement with the formula $H = 2\Phi^2$.

The subsequent evolution of this flux tube configuration is governed by the curvature force acting separately in each flux tube trying to make them contract. Eventually the two tubes come into contact and produce an intense current sheet where they touch. Figure 2 shows the peak current, $\|J\|_\infty$, plotted in two ways,
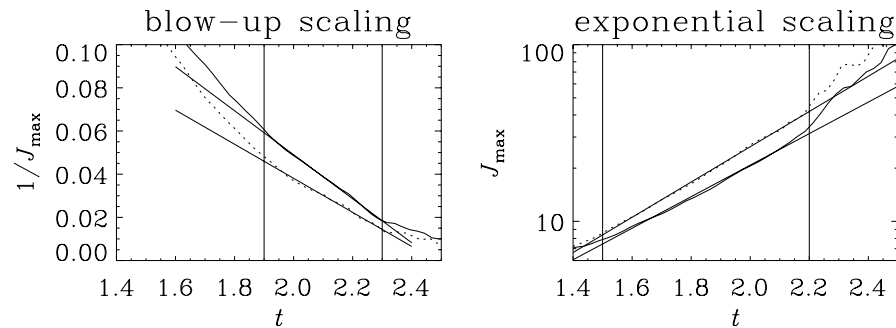
Fig. 2. Semi-logarithmic plot of $\|\boldsymbol{J}\|_\infty$ for a compressible 240^3 calculation in a domain of size 4 (dotted line: filtered, and solid line: unfiltered initial conditions) together with fits to exponential growth and blow-up behavior, respectively. The blow-up scaling fits better at later times.

one showing a period of singular growth and the other showing a period of exponential growth. This initial period is represented in Fig. 1 by the first two frames showing the approach and initial deformation of the linked flux tubes. At the last time visualized, the surfaces in the outer region appear to merge into a single continuous flux tube, while the inner region appears to be annihilated in a complicated reconnected structure with writhe. Figure 3 takes another look at this time using flux lines instead of surfaces. The flux lines in the outer region that appeared to be continuous can now be seen to change direction abruptly where they plunge into the inner region. And the inner region is now seen to be continuously connected to the outer flux lines and instead of being a single flux tubes with writhe, it now appears to be the original flux lines just twisted around each other with almost no reconnection.

In the ideal case, the magnetic helicity is conserved for all time. Even in the resistive case the magnetic helicity is very nearly constant. Furthermore, the peak current increases to large values, which appear to be limited only by the numerical resolution. This is related to the newly posed millennium question of whether regularity of the Navier-Stokes equations can be shown [2]. A singularity probably does not develop for the full viscous and resistive equations due to the development of reconnection. However, singularities do seem possible for Euler and ideal MHD. Numerical calculations have been used to provide insight into the interaction of anti-parallel vortex tubes using the incompressible Euler equations [3]. The key to providing useful results was the direct comparison with hard analytic bounds for the maximum growth rate of the vorticity [4]. This initial condition was very contrived with special symmetries and no helicity, unlike real flows, and its generality remains uncertain.

A similar analytic bound has been shown to exist for the ideal MHD equations [5]. Therefore two intertwined questions have arisen. First, is there an initial condition for ideal MHD which might show similar singular growth? Second, what role might the various types of helicity play in suppressing or enhancing
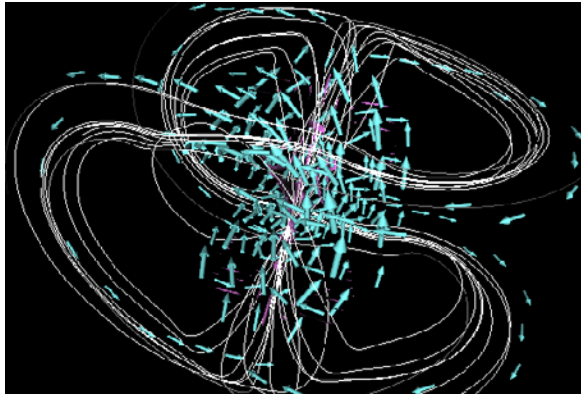
Fig. 3. Magnetic field lines (white) together with some field vectors (in grey) indicating the field orientation for a resistive flux ring calculation at $t = 4$, i.e. shortly after the time of the suspected singularity in the ideal calculation.

the reconnection rate? It has been found [6] that in the ideal case the two inter-locked magnetic flux tubes go through a phase where behavior consistent with a singularity seems plausible. This was surprising because as noted above this is a nearly maximally helical initial condition and was expected to suppress non-linearity. For hydrodynamics, it has also been claimed [7] that the helical initial condition of two orthogonal vortex tubes showed signs of a singularity, although the analytic test [4] was never applied. These two cases raise two possibilities. In hydrodynamics there might exist a mechanism whereby helicity is shed per-mitting stronger nonlinear growth, while in MHD the nonlinearity really comes from $\mathbf{J} \times \mathbf{B}$ and $\mathbf{J} \cdot \mathbf{B}$ is not conserved, so there might in fact be no constraint upon locally strong nonlinearity.

Let us consider an argument for why helicity might be required to allow, at least for a period, nearly singular growth for ideal MHD. This is based upon old arguments for why there could not be a singularity of Euler. It has been argued that a singularity would not occur for Euler because what drives the growth in the vorticity is the axial strain stretching the vorticity, and this strain must grow at the same rate as the vorticity to sustain this growth. This could only be achieved by an enormous growth in the curvature of vortex lines [8], which in turn would require a delicate balance in the growth of the pressure Hessian. This was originally thought not to be feasible, but newer analysis of the anti-parallel Euler calculations [9,10] has shown that all of the analytic requirements needed to achieve this delicate state are in fact obeyed. This is possible because the vorticity and the strain are in fact just different manifestations of the same vector field and can be strongly aligned.

While it has not been shown analytically, one would expect that for ideal MHD to show similar singular growth, a similar delicate balance would have to exist between the vorticity, the current, and the magnetic and velocity strain fields. Current and magnetic strains are just different manifestations of the same

vector magnetic field, just as vorticity and strain are manifestations of the same vector velocity field. However, one is still left with the current being completely distinct from vorticity. Only if there is some property of the vector fields that strongly couples these two fields could they act in concert to give a singularity. Perhaps because helicity is conserved, for strongly helical structures there exists such a constraint. Therefore only for strongly helical magnetic structures could singular, or nearly singular, nonlinear growth occur.

New analysis has shown that the location of the peak in the current is at the juncture between the outer flux lines and the inner flux lines where the maximum in the curvature is located. This would be consistent with new mathematical analysis by J. D. Gibbon (unpublished) that strong magnetic field line curvature should be associated with any singular growth. The vorticity is also the strongest in this region, suggesting the type of symbiotic growth of current and vorticity that we believe is needed if there is to be a singularity of ideal MHD.

Figure 3 is resistive, but is very similar to visualizations of new ideal calculations that were run at higher resolution, up to the equivalent of 1296^3 mesh points if a uniform mesh had been used. However, these new calculations, while they do extend the period of seemingly singular growth, now appear to show that the singular growth eventually is suppressed in the incompressible case. The evidence relies on consistent behavior between the two highest resolution calculations.

What might be the cause of this suppression? While it will take time to fully understand these massive data sets, the initial indications are that it is occurring as the peak in the current moves outside the inner region with its strongly aligned vorticity, magnetic, and current fields. Our suspicion is that the importance of the inner region is that, through twist, the location of most of the initial helicity associated with the linked flux tubes is in the inner region. If this can be shown, then it might tell us that the secret to maintaining nearly singular growth is to maintain as high a level of local helicity for as long as possible. This would be consistent with arguments [11] that fast reconnection is associated with the entanglement of flux lines due to footpoint motion, which is known to produce the required heating rates [12]. Helicity has also been shown to play a role in coronal simulations of an arcade and a twisted flux loop [13], with nearly singular growth in current similar to what we have observed.

There are other possible mechanisms that could suppress singular growth. In a simulation [14] of nearly the same initial condition as the one used in our original paper [6], there is only exponential growth that is associated with the appearance of current sheets. More recent detailed analysis of our calculation shows that the exponential growth is actually associated with the appearance of two nearly overlapping orthogonal current sheets and the pressure barrier between them that suppresses stretching terms and growth. This is an important result because in some sense the more physical initial situation might be two flux tubes that do not overlap at all. Our simulations that show stronger growth in the current all have some overlapping between the initial flux tubes, something
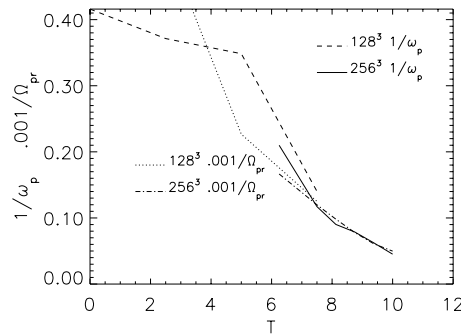
Fig. 4. $1/\|\omega\|_\infty$ and $1/\langle\omega_i e_{ij}\omega_j\rangle$ in Euler for orthogonal vortices.
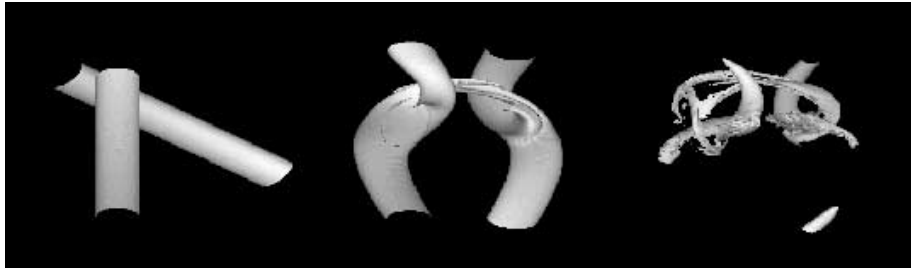

Fig. 5. Isosurfaces of vorticity as a fraction of the peak vorticity. The three frames are $t = 0, 6$ and 10. Arms are pulled out of the original vortices, become anti-parallel, then vorticity within the arms develops singular behavior.

that should not happen in an astrophysical situation where the flux tubes are initially separated by large distances.

4   Orthogonal Vortex Tubes

We now turn to the case of *straight* tubes that are orthogonal to each other. We note that also in this case there is finite helicity. The magnetic case has been studied previously [15], but here we focus on the hydrodynamic case with vortex tubes. Figure 4 shows the inverse of the peak vorticity and the inverse of the enstrophy production rate for the orthogonal vortex tubes whose inviscid evolution is shown in Fig. 5. Plotting these inverses was previously shown to be the most effective way to highlight the $1/(t_c - t)$ singular behavior. Figure 4 shows that the initial growth is weak, unlike the anti-parallel case. Then the growth of peak vorticity, ω_p, and enstrophy production, Ω_{pr}, becomes stronger with their inverses going roughly linearly to zero at the same singular time.

What is the configuration around the peak vorticity once singular growth starts? And what role does helicity play? Analysis of the three-dimensional fields shows that the peak vorticity is located in the arms that are being pulled off

of the two original orthogonal vortices. The last time shows that these isosurfaces are parallel, and analysis shows that the vorticity within these surfaces is anti-parallel. That is, to develop singular growth exactly the same alignment of vorticity that was previously described as a contrived situation is actually what the dynamics generate by themselves. This is consistent with vortex filament work [16]. In terms of helicity, locally around the anti-parallel vortices there is no kinetic helicity density. Therefore in order for orthogonal vortices to develop singular growth, the flow must realign itself to be non-helical, shedding any helicity to achieve this.

In conclusion, these calculations have demonstrated that the role of helicity can be rather complex. In the hydrodynamic case it appears that the absence of helicity is required for there to be singular growth and in the MHD case helicity seems to be required. The role of helicity upon reconnection should now be investigated for these and similar configurations [15].

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