

Spontaneous chiral symmetry breaking in the Tayler instability

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Abstract. The chiral symmetry breaking properties of the Tayler instability are discussed. Effective amplitude equations are determined in one case. This model has three free parameters that are determined numerically. Comparison with chiral symmetry breaking in biochemistry is made.

Keywords. Sun: magnetic fields, dynamo, magnetic helicity

1. Introduction

An important ingredient to the solar dynamo is the α effect. Mathematically speaking α is a pseudo scalar that can be constructed using gravity \mathbf{g} (a polar vector) and angular velocity $\mathbf{\Omega}$ (an axial vector): $\mathbf{g} \cdot \mathbf{\Omega}$ is thus a pseudo scalar and is proportional to $\cos \theta$, where θ is the colatitude. This pseudo scalar changes sign across the equator. This explanation for large-scale astrophysical dynamos works well and therefore one used to think that the existence of the α effect in dynamo theory requires always the existence of a pseudo scalar in the problem. This has indeed been general wisdom, although it has rarely been emphasized in the literature. That this is actually not the case has only recently been emphasized and demonstrated. One example is the magnetic buoyancy instability in the *absence* of rotation, but with a horizontal magnetic field \mathbf{B} and vertical gravity \mathbf{g} being perpendicular to each other, so the pseudo scalar $\mathbf{g} \cdot \mathbf{B}$ vanishes (Chatterjee *et al.* 2011). Another example is the Tayler instability of a purely toroidal field in a cylinder (Tayler 1973; Gellert *et al.* 2011). Thus, the magnetic field is again perpendicular to all possible polar vectors that can be constructed, for example the gradient of the magnetic energy density which points in the radial direction. In both cases, kinetic helicity and a finite α , both of either sign, emerge in the nonlinear stage of the instability. In the first example (Chatterjee *et al.* 2011), the α tensor has been computed using the test-field method. In the second (Gellert *et al.* 2011), the components of the α tensor have been computed using the imposed-field method (see Hubbard *et al.* 2009, for a discussion of possible pitfalls in the nonlinear case).

The purpose of the present paper is to examine spontaneous chiral symmetry breaking in the Tayler instability and to estimate numerically the coefficients governing the underlying amplitude equations. This allows us then to make contact with a system of chemical reactions that can give rise to the same type of spontaneous symmetry breaking.

The connection with chemical systems is of interest because the question of spontaneous symmetry breaking has a long history ever since Pasteur (1853) discovered the preferential handedness of certain organic molecules. The preferential handedness of biomolecules is believed to be the result of a bifurcation event that took place at the origin of life itself (Kondepudi & Nelson 1984; Sandars 2003; Brandenburg *et al.* 2005).

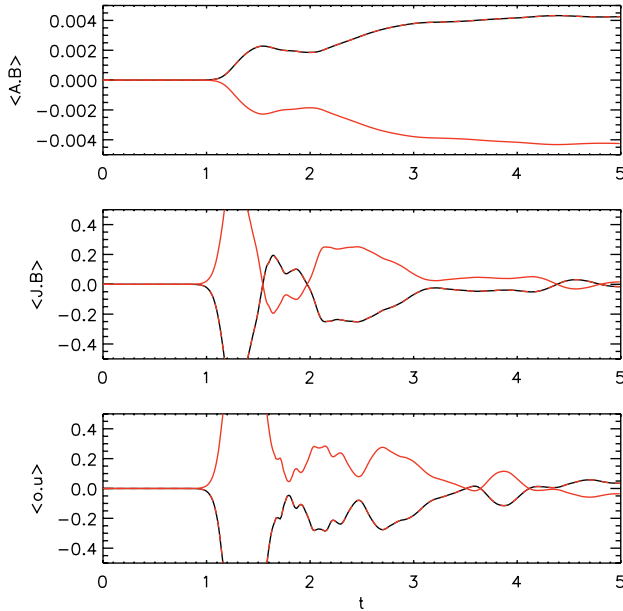


Figure 1. The two lines show the evolution of the volume averaged magnetic helicity $\langle \mathbf{A} \cdot \mathbf{B} \rangle$ for two initial conditions differing only in the parity of their initial perturbations. After the exponential growth magnetic helicity levels off. The inset shows the same in a semi-logarithmic representation. Here, to normalize magnetic helicity and time, we have used the sound speed c_s , $R \equiv s_{\text{in}}$ and B_0 , that is a normalization constant defining the initial magnetic field.

2. Numerical simulations

Our setup consists of an isothermal cylinder with a radial extent from s_{in} to s_{out} and vertical size h . We use cylindrical coordinates s, φ, z and we solve the time dependent ideal MHD equations with periodic boundary conditions in z , reflection in s and periodic in φ and a resolution ranging from 64^3 to 128^3 in the three directions.

The azimuthal field in the basic state is taken of the form

$$B_\varphi = B_0 (s/s_0) \exp[-(s - s_0)^2/\sigma^2]$$

with B_0 being a normalization constant; the axial field B_z is chosen to be zero. In the basic state, the Lorentz force is balanced with a gradient of pressure, and we have checked that our setup was numerically stable if no perturbation was introduced in the system. For the actual calculations we have chosen $h = 2$, $s_{\text{in}} = 1$, $s_{\text{out}} = 3$, $s_0 = 2$ and $\sigma^2 = 0.2$. For a Sun-like star with an average density $\rho \approx 1 \text{g/cm}^3$ the Alfvén travel time is of the order of a year for a 1 kG magnetic field. We therefore expect that everywhere below the surface the sound speed is much greater than the Alfvén speed c_A (Bonanno and Urpin 2011) and we then assume $c_s \geq c_A$, in order to have a sub-thermal magnetic field.

At the beginning of the simulation we perturb the magnetic field. We add a perturbation of amplitude 10^{-7} of the background field. The perturbing field has a given helicity that is either positive or negative. During the development of the instability we observe a net increase of the helicity, as shown in Fig. 1 where we plot time series of the normalized magnetic helicity, which exhibits an initial exponential growth, reaches a peak and then levels off.

3. Amplitude equations

The linear stability analysis of this instability shows that there exist helical growing modes. But the left and right handed modes have exactly the same growth rate independently of their helicity. Hence the growth of helical perturbations cannot be described by a linear theory. However a weakly nonlinear theory is able to describe it as we show below. Let us begin by considering two helical modes of right and left handed varieties, respectively, each of which satisfy the Beltrami relation $\nabla \times \mathbf{R} = \Lambda \mathbf{R}$ and $\nabla \times \mathbf{L} = -\Lambda \mathbf{L}$, where Λ is a coefficient. We can deal with the Fourier transform of these modes,

$$\mathbf{L}(\mathbf{x}) = \int \hat{\mathbf{L}}(\mathbf{q}) d^d q \quad \text{and} \quad \mathbf{R}(\mathbf{x}) = \int \hat{\mathbf{R}}(\mathbf{q}) d^d q \quad (3.1)$$

For the left helical mode, total helicity and energy are given by

$$E_L = \frac{1}{2} \int \mathbf{L}^2(\mathbf{x}) d^d x = \frac{1}{2} \int \hat{\mathbf{L}} \cdot \hat{\mathbf{L}}^* d^d q \quad \text{and} \quad \mathcal{H}_L = \int \mathbf{L} \cdot \nabla \times \mathbf{L} d^d x = -2\Lambda E_L, \quad (3.2)$$

where $*$ denotes complex conjugation. We then have $E = E_L + E_R$ being the total energy and $\mathcal{H} = \mathcal{H}_L + \mathcal{H}_R$ the total helicity. An analogous relation holds also for E_R and \mathcal{H}_R .

In the weakly nonlinear regime the evolution of these modes can be described by general equations of the form:

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} = \frac{\delta \mathcal{L}}{\delta \hat{\mathbf{L}}} \quad \text{and} \quad \frac{\partial \hat{\mathbf{R}}}{\partial t} = \frac{\delta \mathcal{L}}{\delta \hat{\mathbf{R}}}, \quad (3.3)$$

where the form of the Lagrangian \mathcal{L} can be written down from symmetry considerations. In the present case one has to consider the fact that under parity transformation L and R can interchange into each other. With this additional symmetry the simplest Lagrangian takes the following form (Fauve *et al.* 1991)

$$\mathcal{L}[\hat{\mathbf{L}}, \hat{\mathbf{R}}] = \int \left[\gamma \left(|\hat{\mathbf{L}}|^2 + |\hat{\mathbf{R}}|^2 \right) - \mu \left(|\hat{\mathbf{L}}|^4 + |\hat{\mathbf{R}}|^4 - \mu_* |\hat{\mathbf{L}}|^2 |\hat{\mathbf{R}}|^2 \right) \right] d^d q, \quad (3.4)$$

The coefficients γ , μ and μ_* cannot be found from symmetry considerations. Note that in order to show the simplest form, in writing down the Lagrangian we have ignored dissipation. This gives rise to the following set of amplitude equations,

$$\frac{\partial \hat{\mathbf{L}}}{\partial t} = \gamma \hat{\mathbf{L}} - \left(\mu |\hat{\mathbf{L}}|^2 + \mu_* |\hat{\mathbf{R}}|^2 \right) \hat{\mathbf{L}}, \quad \frac{\partial \hat{\mathbf{R}}}{\partial t} = \gamma \hat{\mathbf{R}} - \left(\mu |\hat{\mathbf{R}}|^2 + \mu_* |\hat{\mathbf{L}}|^2 \right) \hat{\mathbf{R}}. \quad (3.5)$$

For certain range of parameters these coupled equations allow the growth of one mode at the expense of the other (Fauve *et al.* 1991), a phenomenon known to biologists by the name ‘‘mutual antagonism’’ (Frank 1953).

Using Eqs. (3.2) and (3.5) and defining $H = \mathcal{H}/2\Lambda$ we obtain equations for E and H ,

$$\frac{dE}{dt} = 2\gamma E - 2(\mu + \mu_*)E^2 - 2(\mu - \mu_*)H^2, \quad (3.6)$$

$$\frac{dH}{dt} = 2\gamma H - 4\mu EH. \quad (3.7)$$

Hence, by calculating the total energy and helicity from direct numerical simulations (DNS) we can determine the unknown coefficients γ , μ and μ_* .

To determine the coefficients γ , μ , and μ_* , we define the instantaneous logarithmic time derivatives of E and H , $\gamma_E = \frac{1}{2} d \ln E / dt$ and $\gamma_H = \frac{1}{2} d \ln H / dt$, so we have

$$\gamma = \gamma_H + 2\mu E, \quad \mu = (\gamma - \gamma_H)/2E, \quad \mu_* = [(\gamma - \gamma_E)E - \mu(E^2 + H^2)]/(E^2 - H^2). \quad (3.8)$$

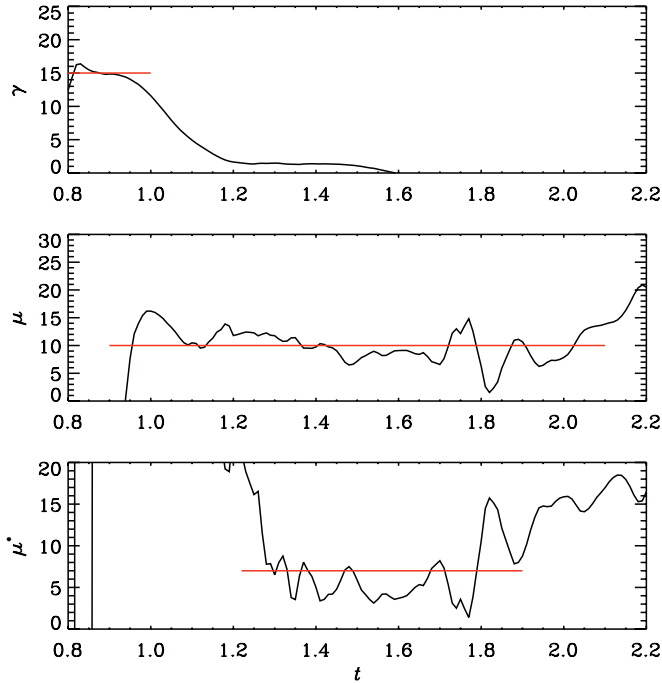


Figure 2. Time dependence of γ , μ , and μ_* , normalized in terms of inner radius and sound speed. The horizontal (red in the online version) lines give the fit results $\gamma R/c_s \approx 14$, $\mu R/c_s^3 \approx 10$, and $\mu^* R/c_s^3 \approx 7$.

The result is shown in Fig. 2, where we can identify first the value of $\gamma R/c_s \approx 14$ during the initial linear growth phase of the instability, and then the values $\mu R/c_s^3 \approx 10$ and $\mu^* R/c_s^3 \approx 7$ during the nonlinear stage. Here R is the inner radius of the cylinder.

4. Conclusions

The present work has demonstrated that the Tayler instability can produce parity-breaking and that it is possible to empirically determine fit parameters that reproduce the nonlinear evolution of energy and helicity. So far, no rigorous derivation of the amplitude equations exists, so this would be an important next step. However, it should be emphasized that chiral symmetry breaking instabilities in biochemistry is described by equations that are identical to those used here; see Bonanno *et al.* (2012) for details.

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Discussion

GUSTAVO GUERRERO: Is this instability also happening in the solar radiative zone?

FABIO DEL SORDO: There are no direct observations of the appearance of this instability in the radiative zone of the Sun. Nevertheless, a toroidal field is likely present in that zone and the Tayler instability, in this case, would play an important role.