ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 1 (Due Wed., Sept 7, 2016) September 19, 2016, Axel Brandenburg

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1. Picking up some pieces.

(a) For a monatomic gas, the stress tensor is $\boldsymbol{\tau} = 2\rho\nu\mathsf{S}$ with $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}$. Show that

$$\boldsymbol{\tau} : \boldsymbol{\nabla u} = 2\rho\nu\mathsf{S}^2. \tag{1}$$

Hint: make use of the facts that (i) S is a symmetric tensor, and (ii) it is trace-free.

(b) Furthermore, show that in this case, the viscous acceleration is

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \rho\nu(\nabla^2\boldsymbol{u} + \tfrac{1}{3}\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u} + 2\mathsf{S}\boldsymbol{\nabla}\ln\rho\nu). \tag{2}$$

(c) Show that, for an ideal gas,

$$\rho T\frac{\mathrm{D}S}{\mathrm{D}t} = \rho c_\mathrm{p}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\mathrm{D}P}{\mathrm{D}t}. \tag{3}$$

Hint: use the facts that $\mathcal{R}/\mu = c_\mathrm{p} - c_\mathrm{v}$, $\mathrm{D}S = c_\mathrm{v}\mathrm{D}\ln P - c_\mathrm{p}\mathrm{D}\ln\rho$, and $\mathrm{D}\ln P = \mathrm{D}\ln T + \mathrm{D}\ln\rho$. Note: there is really a c_p factor in front of the $\mathrm{D}T/\mathrm{D}t$ term, not c_v.

(d) During the lecture, we discussed the isochoric and isobaric instability criteria ($\beta < 0$ and $\beta < 1$, respectively. The latter is the more stringent one. Compare now with the isentropic instability criterion

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_S < 0 \quad \text{(for instability)}. \tag{4}$$

What is the condition on β? Hint: use Eq. (27) from Handout 1 and make sure you differentiate ρ with respect to T such that $S = \text{const}$. To do this, write the density in the form $\rho = \rho(T, S)$.

(a) We split $\boldsymbol{\nabla u}$ into symmetric and antisymmetric parts, i.e.

$$(\boldsymbol{\nabla u})_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) + \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{5}$$

Using the fact that a symmetric matrix multiplied by an antisymmetric one[1] vanishes, we find

$$\boldsymbol{\tau} : \boldsymbol{\nabla u} = 2\rho\nu\mathsf{S}_{ij}\,\tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{6}$$

Furthermore, S_{ij} is trace-free, i.e., $\mathsf{S}_{ij}\delta_{ij} = 0$, so

$$\boldsymbol{\tau} : \boldsymbol{\nabla u} = 2\rho\nu\mathsf{S}_{ij}\left[\tfrac{1}{2}(u_{i,j} + u_{j,i}) + \delta_{ij} \times \text{“(anything)”}\right]. \tag{7}$$

Choosing “(anything)” $= -\frac{1}{3}\boldsymbol{\nabla u}$, we have

$$\boldsymbol{\tau} : \boldsymbol{\nabla u} = 2\rho\nu\mathsf{S}_{ij}\underbrace{[\tfrac{1}{2}(u_{i,j} + u_{j,i}) - \tfrac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}]}_{=\mathsf{S}_{ij}} = 2\rho\nu\mathsf{S}. \tag{8}$$

[1] Note that $\frac{1}{2}(u_{i,j} - u_{j,i}) = -\frac{1}{2}\epsilon_{ijk}\omega_k$, where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity.

(b) Insert $\boldsymbol{\tau} = 2\rho\nu\mathsf{S}$ and use the product rule, so

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_i = (2\rho\nu\mathsf{S}_{ij})_{,j} = 2\nabla_j(\rho\nu)\mathsf{S}_{ij} + 2\rho\nu\mathsf{S}_{ij,j}. \tag{9}$$

Note that

$$\mathsf{S}_{ij,j} = \tfrac{1}{2}(u_{i,jj} + u_{j,ij}) - \tfrac{1}{3}\delta_{ij}(\boldsymbol{\nabla}\cdot\boldsymbol{u})_{,j}. \tag{10}$$

In Cartesian coordinates, $u_{j,ij} = u_{j,ji}$, which is the same as $\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u}$. Furthermore, $\frac{1}{3}\delta_{ij}(\boldsymbol{\nabla}\cdot\boldsymbol{u})_{,j} = \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})_{,i} = \frac{1}{3}(\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u})_i$, so, together with the factor 2, we have $1-2/3=1/3$ times $\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u}$. Also, $u_{i,jj} = (\nabla^2\boldsymbol{u})_i$, so we have

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_i = \rho\nu(\nabla^2\boldsymbol{u})_i + \rho\nu\tfrac{1}{3}(\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u})_i + 2\nabla_j(\rho\nu)\mathsf{S}_{ij}. \tag{11}$$

Finally, writing $\boldsymbol{\nabla}(\rho\nu) = \rho\nu\boldsymbol{\nabla}\ln\rho\nu$, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \rho\nu(\nabla^2\boldsymbol{u} + \tfrac{1}{3}\boldsymbol{\nabla\nabla}\cdot\boldsymbol{u} + 2\mathsf{S}\boldsymbol{\nabla}\ln\rho\nu), \tag{12}$$

where one may think of $\mathsf{S}\boldsymbol{\nabla}\ln\rho\nu$ as a multiplication of a matrix with a vector, which is normally written without a dot. However, this operation does involve a contraction over j, so one could also write this as $\mathsf{S}\cdot\boldsymbol{\nabla}\ln\rho\nu$.

(c) Insert $\mathrm{D}S = c_\mathrm{v}\mathrm{D}\ln P - c_\mathrm{p}\mathrm{D}\ln\rho$, so

$$\rho T\mathrm{D}S = \rho Tc_\mathrm{v}\mathrm{D}\ln P - \rho Tc_\mathrm{p}\mathrm{D}\ln\rho = \rho Tc_\mathrm{v}\mathrm{D}\ln P - \rho Tc_\mathrm{p}\left(\mathrm{D}\ln P - \mathrm{D}\ln T\right), \tag{13}$$

and combine, so that

$$\rho T\mathrm{D}S = \rho c_\mathrm{p}\mathrm{D}T - (c_\mathrm{p}-c_\mathrm{v})\rho T\mathrm{D}\ln P = \rho c_\mathrm{p}\mathrm{D}T - (c_\mathrm{p}-c_\mathrm{v})\frac{\rho T}{P}\mathrm{D}P = \rho c_\mathrm{p}\mathrm{D}T - \mathrm{D}P, \tag{14}$$

where we have made use of the ideal gas equation, i.e., $P = (c_\mathrm{p}-c_\mathrm{v})\rho T$. Thus, we have

$$\rho T\frac{\mathrm{D}S}{\mathrm{D}t} = \rho c_\mathrm{p}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\mathrm{D}P}{\mathrm{D}t}. \tag{15}$$

(d) The isentropic instability criterion

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_S < 0 \quad \text{(for instability)} \tag{16}$$

implies

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_S = \left(\frac{\partial\mathcal{L}}{\partial T}\right)_\rho + \left(\frac{\partial\mathcal{L}}{\partial\rho}\right)_T\left(\frac{\partial\rho}{\partial T}\right)_S < 0 \quad \text{(for instability)}. \tag{17}$$

We now write the density in the form $\rho = \rho(T,S)$ and use $\mathrm{d}S = c_\mathrm{v}\mathrm{d}\ln P - c_\mathrm{p}\mathrm{d}\ln\rho$, and use $\mathrm{d}\ln P = \mathrm{d}\ln\rho + \mathrm{d}\ln T$, so

$$\mathrm{d}S = c_\mathrm{v}\mathrm{d}\ln\rho + c_\mathrm{v}\mathrm{d}\ln T - c_\mathrm{p}\mathrm{d}\ln\rho = c_\mathrm{v}\mathrm{d}\ln T - (c_\mathrm{p}-c_\mathrm{v})\mathrm{d}\ln\rho. \tag{18}$$

To compute $(\partial\rho/\partial T)_S$ we set $\mathrm{d}S=0$ and find

$$\left(\frac{\partial\rho}{\partial T}\right)_S = \frac{\rho}{T}\left(\frac{\partial\ln\rho}{\partial\ln T}\right)_S = \frac{\rho}{T}\frac{c_\mathrm{v}}{c_\mathrm{p}-c_\mathrm{v}} = \frac{1}{\gamma-1}\frac{\rho}{T}, \tag{19}$$

and therefore

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_S = \left(\frac{\partial\mathcal{L}}{\partial T}\right)_\rho + \frac{1}{\gamma-1}\frac{\rho}{T}\left(\frac{\partial\mathcal{L}}{\partial\rho}\right)_T < 0 \quad \text{(for instability)}. \tag{20}$$

Using now $\mathcal{L} = \rho\Lambda(T) - \Gamma$ with $\Lambda(T) = \Lambda_0 T^\beta$ and $\Gamma = \text{const}$, we have $(\partial\mathcal{L}/\partial T)_\rho = \beta\rho\Lambda/T$ and $(\partial\mathcal{L}/\partial\rho)_T = \Lambda$, so

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_S = \beta\rho\Lambda/T + \frac{1}{\gamma-1}\frac{\rho}{T}\Lambda < 0 = \left(\beta + \frac{1}{\gamma-1}\right)\frac{\rho}{T}\Lambda < 0 \quad \text{(for instability).} \tag{21}$$

For $\gamma = 5/3$, we have $\beta < -3/2$ for instability, which is less stringent than both the isochoric and isobaric instability!

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2. Momentum and energy equations in conservative forms. Consider the continuity, momentum, and energy equations in the form

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{22}$$

and

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P = 0, \tag{23}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{P}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{24}$$

where e is the internal energy per unit mass.

(a) Derive the evolution equation for the momentum density

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij}P) \tag{25}$$

Note that summation over double indices is assumed!

(b) Explain why this equation is in *conservative* form. Discuss how the volume-integrated momentum changes for periodic boundary conditions. What other boundary conditions give the same result?

(c) Derive the so-called total energy equation in the form

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j\left(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e + P\right)\right], \tag{26}$$

Again, summation over double indices is assumed.

(d) Explain in words how these equations can be used to say something about hydrodynamic planar shocks, where density, pressure, and velocity can change discontinuously across a surface. Consider a one-dimensional frame of reference *comoving* with the shock. What happens to the time derivative in that frame? Use the equation of state in the form

$$P = (\gamma-1)\rho e$$

and count how many unknowns do you have?

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(a) Using the product rule, we have

$$\frac{\partial\rho u_i}{\partial t} = \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial\rho}{\partial t}. \tag{27}$$

Likewise

$$\frac{\partial}{\partial x_j}[(\rho u_j)u_i] = (\rho u_j)\frac{\partial}{\partial x_j}u_i + u_i\frac{\partial}{\partial x_j}(\rho u_j). \tag{28}$$

With this we can write

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = \rho\frac{\mathrm{D}u_i}{\mathrm{D}t} + u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right). \tag{29}$$

Inserting $\rho\mathrm{D}u_i/\mathrm{D}t = -\partial p/\partial x_i$, and noting that we can write $\partial p/\partial x_i = \partial(\delta_{ij}p)/\partial x_j$, we can pull this term underneath the $\partial/\partial x_j$ derivative and have

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i + \delta_{ij}p) = 0. \tag{30}$$

(b) It is in conservative form, because the rate of change of the momentum $\rho\boldsymbol{u}$ is given by a negative divergence term. Integrating over a certain volume, the rate of change of the integrated momentum is given by the surface integral of the momentum tensor, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho u_i \,\mathrm{d}V = -\oint (\rho u_j u_i + \delta_{ij}p)\,\mathrm{d}S_j. \tag{31}$$

In vanishes for periodic boundary conditions, for example, but it also vanishes if ρ and p vanish on the boundary of the domain. However, requiring ρ and p to vanish on the boundary is unphysical, so in practice it will not be possible to preserve the integrated momentum in the presence of physically meaningful boundaries.

(c) We use the energy and momentum equations,

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = -p\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{32}$$

as well as the momentum equation,

$$\rho u_i\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p. \tag{33}$$

Add the two gives

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho\boldsymbol{u}^2) + \frac{\partial}{\partial x_j}(\rho u_j e + \tfrac{1}{2}\rho u_j\boldsymbol{u}^2) = -\boldsymbol{\nabla}\cdot(p\boldsymbol{u}). \tag{34}$$

Thus, we have

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho\boldsymbol{u}^2) + \frac{\partial}{\partial x_j}(\rho u_j e + \tfrac{1}{2}\rho u_j\boldsymbol{u}^2 + p u_j) = 0, \tag{35}$$

or

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho\boldsymbol{u}^2) + \frac{\partial}{\partial x_j}\left[u_j(\rho e + \tfrac{1}{2}\rho\boldsymbol{u}^2 + p)\right] = 0. \tag{36}$$

(d) In a frame moving with the shock, the shock is stationary and therefore all time derivatives vanish, and therefore the divergences vanish. Since the shock is planar, we can assume it to move along the x direction, so the divergences are just x derivatives, and thus the terms underneath these x derivatives must be constant, i.e., equal when evaluated on both sides of the shock. Thus, we have

$$\rho u_x = \mathrm{const},$$

$$\rho u_x^2 + p = \text{const},$$

$$u_x(\tfrac{1}{2}\rho u_x^2 + \rho e + p) = \text{const}.$$

Together with an equation of state, $p = (\gamma - 1)\rho e$, we have 4 equations for 4 unknowns, Assuming that we know the state of the shock on one side, we can use these 4 equations to solve for the 4 unknowns u_x, ρ, p, and e on the other side of the shock.
Incidently, the last of the three equations can be rewritten as $\rho u_x(\tfrac{1}{2}u_x^2 + e + p/\rho) = \text{const}$, so by using the first equation, the last one can be written as

$$\tfrac{1}{2}u_x^2 + e + p/\rho = \text{const},$$

which is now a *different* constant.

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3. Sound waves in an isothermally stratified atmosphere. Consider the continuity and momentum equations for an isothermal atmosphere (constant temperature) and an isothermal equation of state (special case with $\gamma = 1$) with constant speed of sound, c_s, and uniform gravity, g, in one dimension,

$$\frac{\partial \rho}{\partial t} + u_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial u_z}{\partial z} = 0, \tag{37}$$

$$\rho \frac{\partial u_z}{\partial t} + \rho u_z \frac{\partial u_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \tag{38}$$

where ρ is density and u_z vertical velocity.

(a) Show that these equations obey an equilibrium solution $u_z = u_{z0}(z)$, $\rho = \rho_0(z)$, given by

$$u_{z0}(z) = 0, \quad \rho_0(z) = \rho_{00}\, e^{-z/H}, \tag{39}$$

where ρ_{00} is a constant and $H = c_s^2/g$ is the vertical scale height.

(b) Write $\rho = \rho_0 + \rho_1$ and $u_z = u_{z1}$ and linearize equations (37) and (38) with respect to ρ_1 and u_{z1}.

(c) Assume that ρ_1 and u_{z1} take the form

$$\rho_1(z,t) = \hat{\rho}_1 e^{\mathrm{i}kz - \mathrm{i}\omega t - z/2H}, \tag{40}$$

$$u_{z1}(z,t) = \hat{u}_{z1} e^{\mathrm{i}kz - \mathrm{i}\omega t + z/2H}, \tag{41}$$

and show that the linearized equations can be written as

$$\begin{pmatrix} -\mathrm{i}\omega & [\mathrm{i}k - (2H)^{-1}] \\ [\mathrm{i}k + (2H)^{-1}]c_s^2 & -\mathrm{i}\omega \end{pmatrix} \begin{pmatrix} \hat{\rho}_1 \\ \rho_{00}\hat{u}_{z1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{42}$$

(d) Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.

(e) Give a qualitative plot of the dispersion relation.

(f) Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6\,\mathrm{km/s}$ and $g = 270\,\mathrm{m/s^2}$, and the Earth's atmosphere, assuming $c_s = 300\,\mathrm{m/s}$ and $g = 10\,\mathrm{m/s^2}$.

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(a) In hydrostatic equilibrium we have

$$c_s^2 \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}z} = -g,$$

so $\ln(\rho_0/\rho_{00}) = -gz/c_s^2$ and therefore $\rho_0 = \rho_{00}\exp(-gz/c_s^2)$, which we write as $\rho_0 = \rho_{00}\exp(-z/H)$, where $H = c_s^2/g$ is the scale height.

(b) The linearized equations take the form

$$\frac{\partial\rho_1}{\partial t} + u_{z1}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} + \rho_0\frac{\partial u_{z1}}{\partial z} = 0, \tag{43}$$

$$\rho_0\frac{\partial u_{z1}}{\partial t} + c_s^2\frac{\partial\rho_1}{\partial z} + \rho_1 g = 0, \tag{44}$$

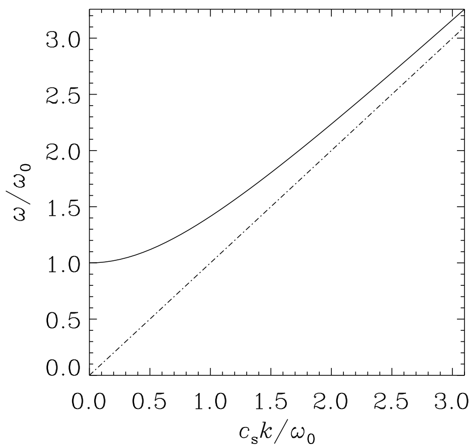

Figure 1: Dispersion relation.

(c) Inserting Eqs. (40) and (41), we have

$$-\mathrm{i}\omega\hat\rho_1 e^{\mathrm{i}kz-\mathrm{i}\omega t-z/2H} + \hat u_{z1}e^{\mathrm{i}kz-\mathrm{i}\omega t+z/2H}\left(-\frac{\rho_0}{H}e^{-z/H}\right) + \rho_0 e^{-z/H}\left(\mathrm{i}k+\frac{1}{2H}\right)\hat u_{z1}e^{\mathrm{i}kz-\mathrm{i}\omega t+z/2H} = 0, \tag{45}$$

$$-\mathrm{i}\omega\rho_0 e^{-z/H}\hat u_{z1}e^{\mathrm{i}kz-\mathrm{i}\omega t+z/2H} + c_s^2\left(\mathrm{i}k-\frac{1}{2H}\right)\hat\rho_1 e^{\mathrm{i}kz-\mathrm{i}\omega t-z/2H} + g\hat\rho_1 e^{\mathrm{i}kz-\mathrm{i}\omega t-z/2H} = 0. \tag{46}$$

Note that in both equations the exponential factors cancel, which requires in some expressions the presence of the $e^{-z/H}$ factors from the background density. Thus, we have

$$-\mathrm{i}\omega\hat\rho_1 + \hat u_{z1}\left(-\frac{\rho_0}{H}\right) + \rho_0\left(\mathrm{i}k+\frac{1}{2H}\right)\hat u_{z1} = 0, \tag{47}$$

$$-\mathrm{i}\omega\rho_0\hat u_{z1} + c_s^2\left(\mathrm{i}k-\frac{1}{2H}\right)\hat\rho_1 + g\hat\rho_1 = 0, \tag{48}$$

using $g = c_s^2/H$, and combining terms, we have

$$-\mathrm{i}\omega\hat\rho_1 + \left(\mathrm{i}k-\frac{1}{2H}\right)\rho_0\hat u_{z1} = 0, \tag{49}$$

$$-\mathrm{i}\omega\rho_0\hat u_{z1} + c_s^2\left(\mathrm{i}k+\frac{1}{2H}\right)\hat\rho_1 = 0. \tag{50}$$

In matrix form, this can be written as

$$\begin{pmatrix} -\mathrm{i}\omega & [\mathrm{i}k-(2H)^{-1}] \\ [\mathrm{i}k+(2H)^{-1}]c_s^2 & -\mathrm{i}\omega \end{pmatrix}\begin{pmatrix}\hat\rho_1 \\ \rho_0\hat u_{z1}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{51}$$

(d) The determinant of the matrix vanishes when

$$-\omega^2 - \left(-k^2 - \frac{1}{4H^2}\right)c_s^2 = 0, \tag{52}$$

or

$$\omega^2 = c_s^2k^2 + \omega_0^2. \tag{53}$$

(e) Fig. 1 shows two graphic representations of the dispersion relation.

(f) Inserting the numerical values, we have $\omega_0 = c_s/2H = g/2c_s = 270/12{,}000\,\mathrm{s}^{-1} = 0.0225\,\mathrm{s}^{-1}$, so the period is $\approx 280\,\mathrm{s} = 4.7\,\mathrm{min}$. For the Earth's atmosphere, we have $\omega_0 = g/2c_s = 10/600\,\mathrm{s}^{-1} = 0.017\,\mathrm{s}^{-1}$, so the period is $\approx 380\,\mathrm{s} = 6\,\mathrm{min}$.