Problems

1. Consider a one-dimensional shock. Use the ideal fluid equations in conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0,$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e) + \frac{\partial}{\partial x}\left[v\left(\tfrac{1}{2}\rho v^2 + \rho e + p\right)\right] = 0,$$

where e is the internal energy density per unit mass, and the other variables have their usual meaning. Assume a perfect gas with

$$p = (\gamma - 1)\rho e.$$

(a) Why is it useful to consider a frame of reference comoving with the shock? Show that in a frame comoving with the shock the following three quantities are conserved:

$$J = \rho v; \tag{1}$$

$$I = \rho v^2 + p; \tag{2}$$

$$E = \tfrac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}. \tag{3}$$

(b) Eliminate first p/ρ and then ρ, to show that

$$\frac{v_2}{v_1} = \frac{2\gamma}{\gamma + 1}\left(1 + \frac{p_1}{\rho_1 v_1^2}\right) - 1$$

where the subscripts 1 and 2 refer, respectively, to the upstream and downstream sides of the shock.

(c) Calculate v_2 for $v_1 = 5$, $\rho_1 = p_1 = 1$ and $\gamma = 5/3$.

(d) Calculate ρ_2 and p_2. Sketch the velocity and density profiles indicating the positions of the upstream and downstream sides.

(e) State whether the normalised entropy,

$$s = \frac{1}{\gamma}\ln p - \ln \rho$$

is increased or decreased behind the shock. Calculate s_1 and s_2.

[30 marks]

2. Use the PENCIL CODE to solve nonlinear sound waves using as initial conditions

$$u = A \sin kx,$$

$$\ln \rho = B \sin kx,$$

(a) Change the amplitude factors for density and velocity in start.in to make the wave traveling forward or backward. Monitor the wave, e.g., in idl with `.r pvid`.

(b) Increase the amplitude of a traveling wave solution to observe the development of a shock. Increase the viscosity to avoid wiggles. Check that mass and energy are conserved.

(c) Change the order of the scheme (itorder=2 or 1), to find out the error in energy conservation. You might need to adjust the length of the time step by hand (set dt=1e-5 or something).

(d) Use bigger resolutions and consider a Mach number of 10, i.e., choose $A = B = 10$. In that case, use `iheatcond='chi-const'` together with `chi_t=1` and `nu=1` for 512. How well is total energy conserved and how does it change if you use 1024 mesh points?

[**30 marks**]