Astrophysical Fluid Dynamics: Solutions to Problem Sheet 2

Consider a one-dimensional shock. Use the ideal fluid equations in conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0,$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e) + \frac{\partial}{\partial x}\left[v\left(\tfrac{1}{2}\rho v^2 + \rho e + p\right)\right] = 0,$$

where e is the internal energy density per unit mass, and the other variables have their usual meaning. Assume a perfect gas with

$$p = (\gamma - 1)\rho e.$$

1. Why is it useful to consider a frame of reference comoving with the shock? Show that in a frame comoving with the shock the following three quantities are conserved:

$$J = \rho v; \tag{1}$$

$$I = \rho v^2 + p; \tag{2}$$

$$E = \tfrac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}. \tag{3}$$

2. Eliminate first p/ρ and then ρ, to show that

$$\frac{v_2}{v_1} = \frac{2\gamma}{\gamma + 1}\left(1 + \frac{p_1}{\rho_1 v_1^2}\right) - 1$$

   where the subscripts 1 and 2 refer, respectively, to the upstream and downstream sides of the shock.

3. Calculate v_2 for $v_1 = 5$, $\rho_1 = p_1 = 1$ and $\gamma = 5/3$.

4. Calculate ρ_2 and p_2. Sketch the velocity and density profiles indicating the positions of the upstream and downstream sides.

5. State whether the normalised entropy,

$$s = \frac{1}{\gamma}\ln p - \ln \rho$$

   is increased or decreased behind the shock. Calculate s_1 and s_2.

- Answer

1. In a comoving frame of reference, the solution is statitionary, so we can seek solutions for which $\partial/\partial t = 0$. This means that $\rho v =$ const, $\rho v^2 + p =$ const, and

$$v\left(\tfrac{1}{2}\rho v^2 + \rho e + p\right) = \rho v\left(\tfrac{1}{2}v^2 + e + \frac{p}{\rho}\right) = \text{const.}$$

   Since ρv itself is constant, the second terms in parenthesis must also be constant. Also, we can replace $e = (p/\rho)/(\gamma - 1)$. Putting $e + p/\rho$ over the same denominator, we find $\gamma/(\gamma - 1)$ times $e = p/\rho$, so we obtain Eq. (3) above. The constants J, I, and E are determined from the values of ρ, v, and p on the upstream side of the flow.

2. Solve (2) for p/ρ, so we get

$$p/\rho = I/\rho - v^2,$$

so we have

$$E = \tfrac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\left(\frac{I}{\rho} - v^2\right)$$

and thus

$$E = \frac{\gamma}{\gamma-1}\,\frac{I}{\rho} + \left(\tfrac{1}{2} - \frac{\gamma}{\gamma-1}\right)v^2$$

Furthermore, $\rho v = J$ to eliminate ρ, so

$$E = \frac{\gamma}{\gamma-1}\,\frac{I}{J}v + \left(\tfrac{1}{2} - \frac{\gamma}{\gamma-1}\right)v^2$$

Multiply by $2(\gamma-1)/(\gamma+1)$, so

$$v^2 - \frac{2\gamma}{\gamma+1}\,\frac{I}{J}\,v + 2\,\frac{\gamma-1}{\gamma+1}\,E = 0.$$

For any quadratic equation, written as $0 = (v-v_1)(v-v_2) = v^2 - (v_1+v_2)v + v_1v_2$, with solutions v_1 and v_2, we know that the middle term is v_1+v_2, so here we have

$$v_1 + v_2 = \frac{2\gamma}{\gamma+1}\,\frac{I}{J}.$$

Note also that from (1) and (2) we obtain $I/Jv = 1 + p/\rho v^2$. Applied to the upstream we have $I/Jv_1 = 1 + p_1/\rho_1 v_1^2$, so we find

$$\frac{v_2}{v_1} = \frac{2\gamma}{\gamma+1}\left(1 + \frac{p_1}{\rho_1 v_1^2}\right) - 1$$

Use the Pencil Code to solve nonlinear sound waves using as initial conditions

$$u = A\sin kx,$$

$$\ln\rho = B\sin kx,$$

1. Change the amplitude factors for density and velocity in start.in to make the wave traveling forward or backward. Monitor the wave, e.g., in idl with `.r pvid`.
2. Increase the amplitude of a traveling wave solution to observe the development of a shock. Increase the viscosity to avoid wiggles. Check that mass and energy are conserved.
3. Change the order of the scheme (itorder=2 or 1), to find out the error in energy conservation. You might need to adjust the length of the time step by hand (set dt=1e-5 or something).
4. Use bigger resolutions and consider a Mach number of 10, i.e., choose $A = B = 10$. In that case, use `iheatcond='chi-const'` together with `chi_t=1` and `nu=1` for 512. How well is total energy conserved and how does it change if you use 1024 mesh points?

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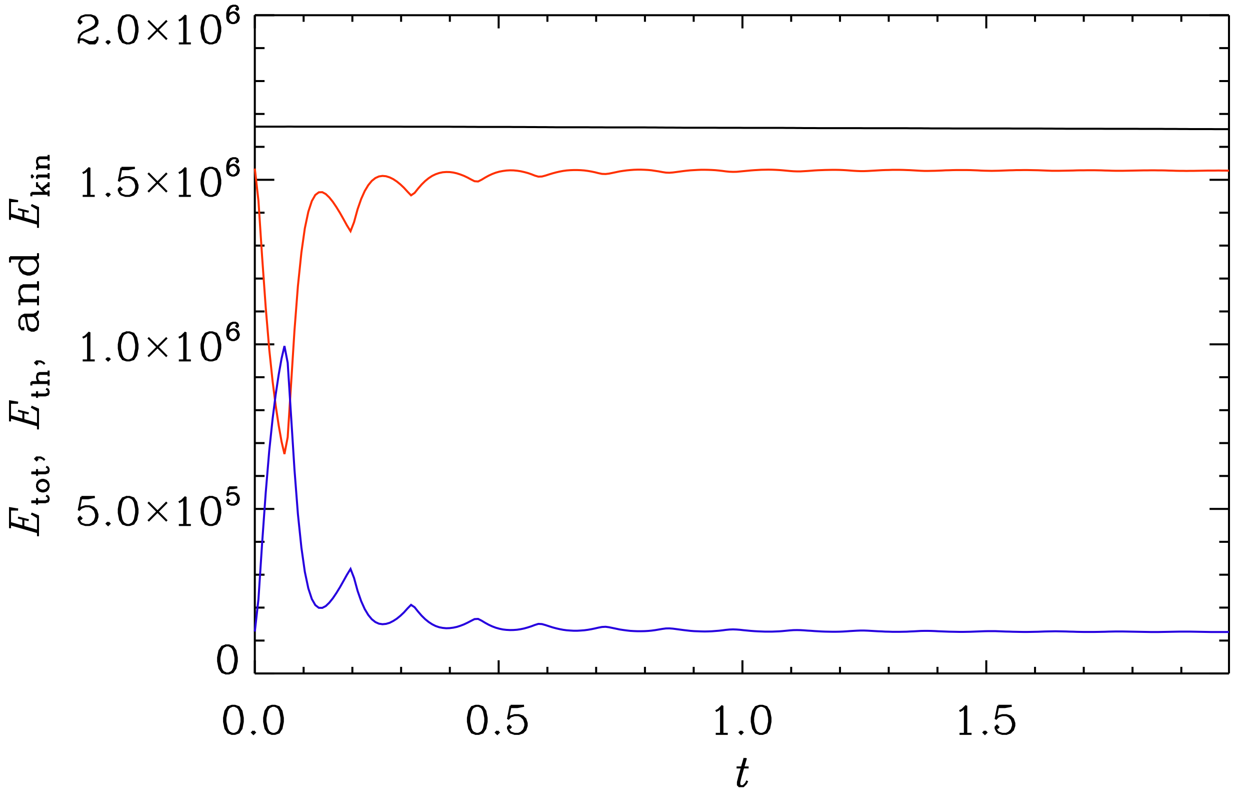

Figure 1: Time evolution of $E_{\rm tot} = E_{\rm th} + E_{\rm kin}$.