Test problem with Eddington approximation

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1 Radiative transfer

The equation of radiative transfer is

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} I = -\kappa\rho\,(I - S). \tag{1}$$

where $I = I(\boldsymbol{x}, t, \hat{\boldsymbol{n}}, \nu)$ is the intensity, ρ is the density, κ is the intensity, and $S = (\sigma_{\rm SB}/\pi)\,T^4$ is the source function.

Equation (1) can be solved by taking moments. We define 0th, 1st, and 2nd moments as follows,

$$J = \frac{1}{4\pi}\int_{4\pi} I\;{\rm d}\Omega, \tag{2}$$

$$\boldsymbol{F} = \int_{4\pi} I\hat{\boldsymbol{n}}\;{\rm d}\Omega, \tag{3}$$

$$\mathsf{P} = \frac{1}{4\pi}\int_{4\pi} I\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\;{\rm d}\Omega. \tag{4}$$

Here, J is the mean intensity, $\boldsymbol{F}$ is the radiative flux, and P is the radiation pressure tensor. Taking the 0th and 1st moment of Equation (1) yields

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = -4\pi\kappa\rho\,(J - S), \tag{5}$$

$$\boldsymbol{\nabla} \cdot \mathsf{P} = -\frac{\kappa\rho}{4\pi}\boldsymbol{F}. \tag{6}$$

Making the closure assumption

$$\mathsf{P}_{ij} = \tfrac{1}{3}\delta_{ij} J. \tag{7}$$

Equation (6) becomes

$$\tfrac{1}{3}\boldsymbol{\nabla} J = -\frac{\kappa\rho}{4\pi}\boldsymbol{F}. \tag{8}$$

Next, to calculate $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, we divide first by $\kappa\rho$ and then take the divergence, so

$$\boldsymbol{\nabla} \cdot \left(\frac{1}{3\kappa\rho}\boldsymbol{\nabla} J\right) = -\frac{1}{4\pi}\boldsymbol{\nabla} \cdot \boldsymbol{F}. \tag{9}$$

Using Equation (5) we obtain a closed equation for J.

$$\boldsymbol{\nabla} \cdot \left(\frac{1}{3\kappa\rho}\boldsymbol{\nabla} J\right) = \kappa\rho\,(J - S). \tag{10}$$

2 Basic hydro equation

We are interested in steady solutions to the time-dependent equations

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla} \cdot \rho\boldsymbol{u} \tag{11}$$

$$\rho\frac{{\rm D}u}{{\rm D}t} = \boldsymbol{\nabla} P + \rho\boldsymbol{g} + \text{visc force} \tag{12}$$

$$\left(\rho T\frac{{\rm D}S}{{\rm D}t} =\right) \quad \rho\frac{{\rm D}E}{{\rm D}t} + P\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\boldsymbol{\nabla} \cdot \boldsymbol{F} \tag{13}$$

where $E = c_{\rm v}T$ is the internal energy with $c_{\rm v} =$ const. Using Equation (8), we have

$$\boldsymbol{F} = -\frac{4\pi}{3\kappa\rho}\boldsymbol{\nabla} J, \tag{14}$$

or, more directly using Equation (5)

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = -\boldsymbol{\nabla} \cdot \frac{4\pi}{3\kappa\rho}\boldsymbol{\nabla} J. \tag{15}$$

3 Equilibrium solution

Using Equation (9), the equilibrium solution, $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, satisfies $J = S = (\sigma_{\rm SB}/\pi)\,T^4$, and therefore

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{F} = -\boldsymbol{\nabla} \cdot \frac{4\sigma_{\rm SB}}{3\kappa\rho}\boldsymbol{\nabla} T^4 = -\boldsymbol{\nabla} \cdot K\boldsymbol{\nabla} T \tag{16}$$

where

$$K = \frac{16\sigma_{\rm SB}T^3}{3\kappa\rho}, \tag{17}$$

so $K{\rm d}T/{\rm d}z = $ const for the equilibrium solution. In the following, we use

$$\kappa = \kappa_0(\rho/\rho_0)^a (T/T_0)^b, \tag{18}$$

where a and b are adjustable parameters, ρ_0 and T_0 are reference values for density and temperature, respectively, and κ_0 gives the overall magnitude of the opacity.

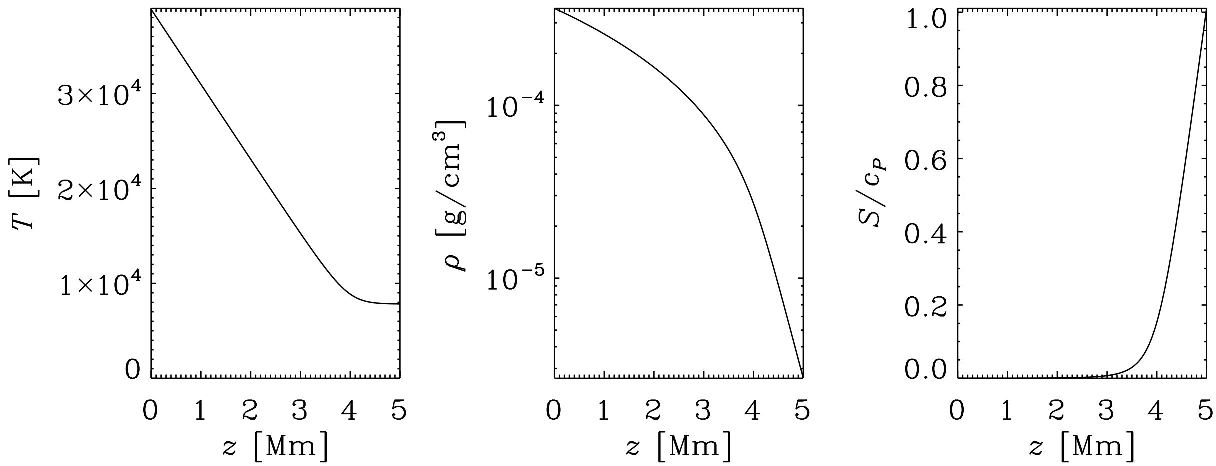

Figure 1: Solution to Equations (19) and (21).

An analytic solution that also satisfies hydrostatic equilibrium is given by [see Equation (68) of Brandenburg (2016)]

$$\frac{T}{T_0} = \left[(n+1)\nabla_{\rm rad}^{(0)}\left(\frac{P}{P_0}\right)^{1+a} + \left(\frac{T_{\rm top}}{T_0}\right)^m\right]^{1/m}, \tag{19}$$

where $m = 4 + a - b$, $\nabla_{\rm rad}^{(0)} = F_{\rm rad}P_0/(K_0T_0\rho_0 g)$, $K_0 = 16\sigma_{\rm SB}T_0^3/(3\kappa_0\rho_0)$ are constants, and $T_{\rm top} = T_{\rm eff}/2^{1/4}$ is an integration constant that is specified such that $T \to T_{\rm top}$ as $P \to 0$. Note also that $4 + a - b = (n + 1)(1 + a)$, where

$$n = (3 - b)/(1 + a) \tag{20}$$

is the polytropic index, so the ratio of $4 + a - b$ to $1 + a$ is just $n + 1$, which enters in front of the $\nabla_{\rm rad}^{(0)}$ term in Equation (19). Since $K \propto T^{3-b}/\rho^{1+a} \propto T^{4+a-b}/P^{1+a}$, we have $K \to$ const $= K_0$ for $T \gg T_{\rm top}$. Finally, the z coordinate is obtained by numerical integration,

$$z = -\int \frac{{\rm d}P}{\rho g} = -\int \frac{P}{\rho g}\,{\rm d}\ln P. \tag{21}$$

4 Example of BB14

Consider, as a specific example, Run B6 of Barekat & Brandenburg (2014) with $a = 1$, $b = 0$, $\kappa_0 = 10^6\,{\rm Mm}^{-1}\,{\rm cm}^3\,{\rm g}^{-1}$ and $T_{\rm eff} = 9300\,{\rm K}$. where $\rho_0 = 4\ 10^{-4}$ g cm^{-3} and $T_0 = 38,968\,{\rm K}$ are used. The result is plotted in Fig. 1.

References

Barekat, A., & Brandenburg, A.: 2014, "Near-polytropic stellar simulations with a radiative surface," *Astron. Astrophys.* **571**, A68

Brandenburg A.: 2016, "Stellar mixing length theory with entropy rain," *Astrophys. J.* **832**, 6

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