TIME DISCRETISATION: Magnetic fields - integrating factor

We can solve the horizontal part of the diffusion in $k$-space by an integrating factor.

The advantage of this is that it is exact and forces more diffusion at higher wavenumbers unlike many other schemes (e.g. Crank-Nicolson which has a wavelike attenuation at the highest wavenumbers in its rational polynomial approximation to the exponential).

\[
\begin{align*}
\partial_t P &= F + C_k \zeta \left( \nabla_h^2 P + \partial_z^2 P \right), \\
\partial_t P + C_k \zeta k^2 P &= F + C_k \zeta \partial_z^2 P,
\end{align*}
\]

where $P$ is $P(k)$ in $k$-space and

\[
F = \frac{(\text{nonlinear})}{k^2}.
\]

So then

\[
\begin{align*}
e^{C_k \zeta k^2 t} (\partial_t P + C_k \zeta k^2 P) &= e^{C_k \zeta k^2 t} (F + C_k \zeta \partial_z^2 P), \\
\partial_t (e^{C_k \zeta k^2 t} P) &= e^{C_k \zeta k^2 t} (F + C_k \zeta \partial_z^2 P),
\end{align*}
\]

Time discretisation (AB3 for $F$, Euler for diffusion) gives

\[
P^{n+1} E(t^{n+1}) - P^n E(t^n) = c_0 F^n E(t^n) + c_1 F^{n-1} E(t^{n-1}) + c_2 F^{n-2} E(t^{n-2}) + \delta t_0 C_k \zeta \partial_z^2 P^n
\]

where $E(t) = e^{C_k \zeta k^2 t}$, so

\[
P^{n+1} = E(-\delta t_0) \left( P^n + c_0 F^n + c_1 F^{n-1} E(-\delta t_1) + c_2 F^{n-2} E(-\delta t_1 - \delta t_2) + \delta t_0 C_k \zeta \partial_z^2 P^n \right)
\]

where $\delta t_0$ is the timestep computed for the next step $n \rightarrow n + 1$, $\delta t_1$ is the timestep for the previous step $n - 1 \rightarrow n$, and $\delta t_2$ is the timestep for the step before that $n - 2 \rightarrow n - 1$.

Disadvantage: have to calculate exponentials – can be expensive