ASTR-3760: Solar and Space Physics . . . . **Problem Set 1** KEY (Due Mon., January 25, 2016)

Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

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1. K index and relation to *B* field. When googling for Kp index, I came across the following site:

`http://www.spaceweatherlive.com/en/help/the-kiruna-magnetometer`

It relates the K index (from 0 to 9) to ranges of the magnetic field ($B_{\rm min}$ and $B_{\rm max}$) in nT; see below.

Interpreting the K-index based on values from Kiruna

De K-index is just like the Kp-index, a geomagnetic storm index with a logarithmic scale from 1 to 9 but as measured by a single station and not from multiple stations combined. Based on the deflection from the Kiruna magnetometer we can try to determine the K-indice for that specific station. For the station at Kiruna, we do this with the help of the table below. Be aware that, due to it's location, this magnetometer is only be helpful for observers from Europe.

K-index	Deflection in nanoTesla	Storm type
0	0 - 15	Quiet conditions
1	15 - 30	Quiet conditions
2	30 - 60	Quiet conditions
3	60 - 120	Unsettled geomagnetic conditions
4	120 - 210	Active geomagnetic conditions
5	210 - 360	G1 - Minor geomagnetic storm
6	360 - 600	G2 - Moderate geomagnetic storm
7	600 - 990	G3 - Strong geomagnetic storm
8	990 - 1500	G4 - Severe geomagnetic storm
9	1500 and more	G5 - Extreme geomagnetic storm

(a) Plot $B_{\rm min}$ and $B_{\rm max}$ versus Kp (either with the computer or by hand) and check that K does indeed grow logarithmically, and thus $B_{\rm min}$ and $B_{\rm max}$ grow exponentially with Kp.

(b) Try to describe the data with the formula

$$B = B_0 \exp(\mathrm{Kp}/\mathrm{Kp}_0)$$

and give the values B_0 and Kp_0. (Don't forget the give units, when appropriate and necessary.)

(c) To see the logarithmic behavior of Kp, it is more natural to plot Kp versus $B_{\rm min}$ and versus $B_{\rm max}$ (in the same graph). Check that

$$\mathrm{Kp} = \mathrm{Kp}_0 \ln(B/B_0)$$

. . . .

(a) To show that $B_{\rm min}$ and $B_{\rm max}$ grow exponentially with Kp, it is best to plot $\log B_{\rm min}$ and $\log B_{\rm max}$ versus Kp. This is shown in Fig. 1. One can now overplot $\log B = \log B_0 \exp(\mathrm{Kp}/\mathrm{Kp}_0)$ for some trial values of B_0 and Kp_0. Alternatively, one can use a semi-logarithmic plot; see Fig. 3, where I plot $B_{\rm min}$ (in blue) and versus $B_{\rm max}$ (in red) vs. Kp on the left and Kp versus $B_{\rm min}$ (in blue) and versus $B_{\rm max}$ (in red) on the right. The straight lines denote approximate fits discussed below.

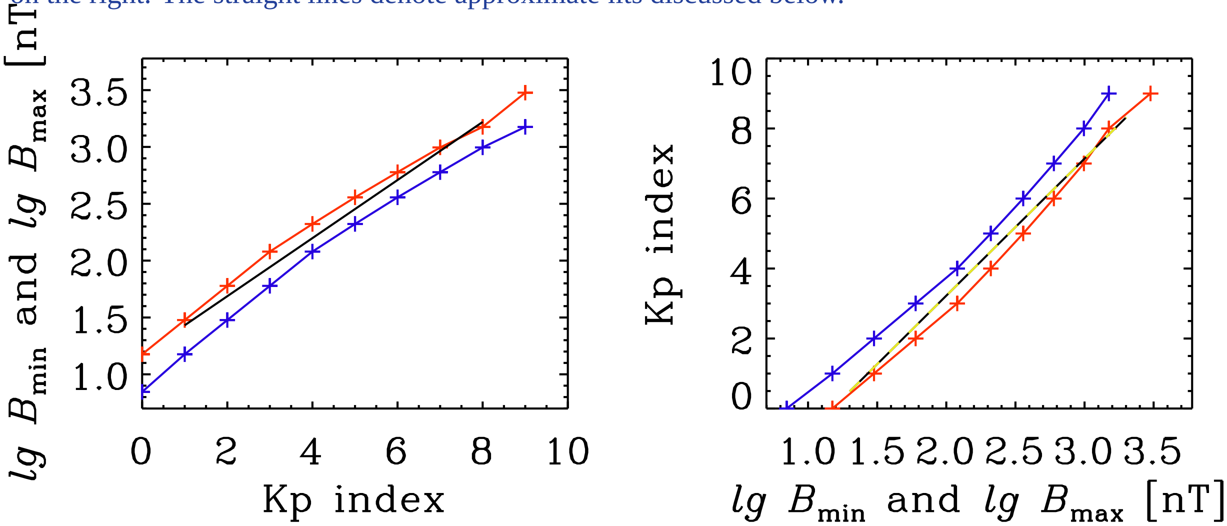

Figure 1: $\log B_{\rm min}$ and $\log B_{\rm max}$ versus Kp (left) and Kp versus $\log B_{\rm min}$ and $\log B_{\rm max}$ (right). (One is essentially just the mirror image of the other.)

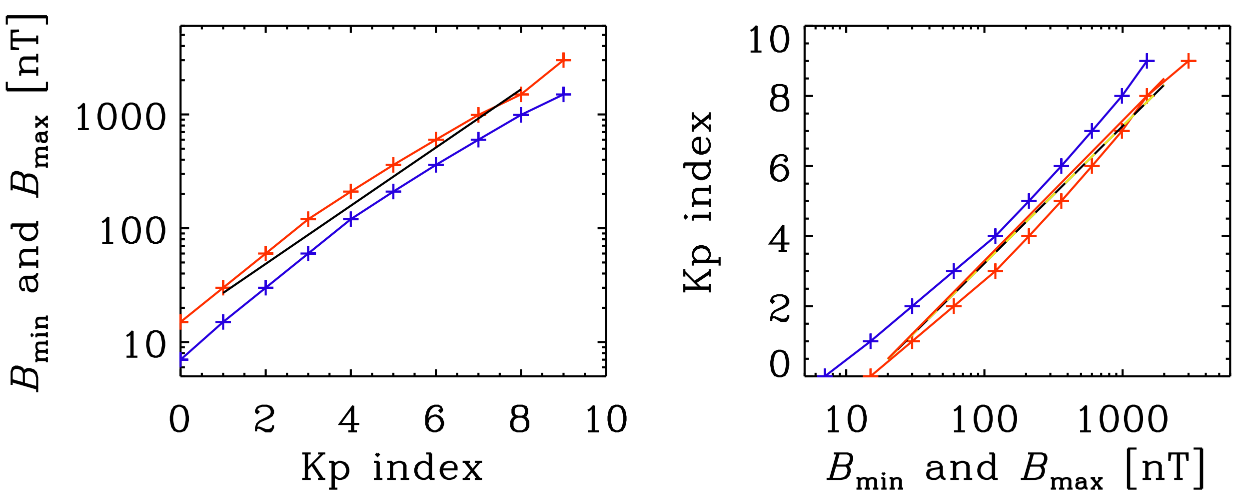

Figure 2: Kp index

(b) After some fiddling, I found that

$$B = B_0 \exp(\mathrm{Kp}/\mathrm{Kp}_0)$$

with $B_0 = 15\,\mathrm{nT}$ and $\mathrm{Kp}_0 = 1.7$ fits the blue and red lines reasonably well (at least in the range $3 \le \mathrm{Kp} \le 6$).

(c) $\tilde{K}_0 = 3.9$, but $\tilde{K}_0 = 4.0$ is also good and even slightly better.

2. Vector & scalar fields. Consider the following vector function in Cartesian space:

$$\mathbf{F} = (y+2xy)\hat{\mathbf{e}}_x + (x+x^2+3y^2z^2)\hat{\mathbf{e}}_y + (2y^3z)\hat{\mathbf{e}}_z$$

(a) Calculate the divergence and curl of **F**.

(b) Can **F** be an electric field? If so, under what circumstances?

(c) Can **F** be a magnetic field? If so, under what circumstances?

(d) Extra credit: Using a table of vector identities, it is possible for you to determine how the vector field **F** was "generated" from a simpler scalar function $\phi(x,y,z)$. Figure out how that was done, and calculate the functional form of $\phi(x,y,z)$.

(a) The divergence is given by:

$$\nabla \cdot \mathbf{F} = 2y + 6yz^2 + 2y^3 = 2y(1+3z^2+y^2) \, .$$

The curl of **F** is zero! If you didn't get that answer, see the following breakdown:

$$\begin{aligned}
(\nabla \times \mathbf{F})_x = (\partial F_z/\partial y) - (\partial F_y/\partial z) &= 6y^2z - 6y^2z = 0 \\
(\nabla \times \mathbf{F})_y = (\partial F_x/\partial z) - (\partial F_z/\partial x) &= 0 - 0 = 0 \\
(\nabla \times \mathbf{F})_z = (\partial F_y/\partial x) - (\partial F_x/\partial y) &= (1+2x) - (1+2x) = 0
\end{aligned}$$

(b) Yes. Its divergence describes the distribution of charge density ρ_c in space. Its curl is zero, which from Faraday's law means that the local magnetic field **B** must be constant in time (i.e., $\partial \mathbf{B}/\partial t = 0$).

(c) No, because its divergence is not zero. A magnetic field must obey $\nabla \cdot \mathbf{B} = 0$.

(d) We know that $\nabla \times \mathbf{F} = 0$, and there is a vector identity that says the curl of a gradient is also identically zero:

$$\nabla \times \nabla \phi = 0 \, .$$

Thus, the vector **F** must be expressible as the gradient of a scalar "potential" function $\phi(x,y,z)$. Using your knowledge of derivatives, it's possible to work backwards from the three components of **F** to see that the potential must have been

$$-\phi(x,y,z) = xy + x^2y + y^3z^2 \, .$$

I've used here a minus sign to conform with the usual convention in which the electric field is the *negative* gradient of ϕ, i.e., $\mathbf{E} = -\nabla\phi$. Thus, **E** in the direction in which ϕ becomes most negative.

3. A Not-So-Ordinary Differential Equation. Consider a one-dimensional "slab" of gas that starts at $x = 0$ and ends at $x = D$, and is surrounded by empty space. A ray of light with intensity I_0 hits the slab at $x = 0$ and shines through it parallel to the x axis. Inside the slab, the intensity obeys

$$\frac{dI}{dx} = \alpha(S - I)$$

where α and S are constants.

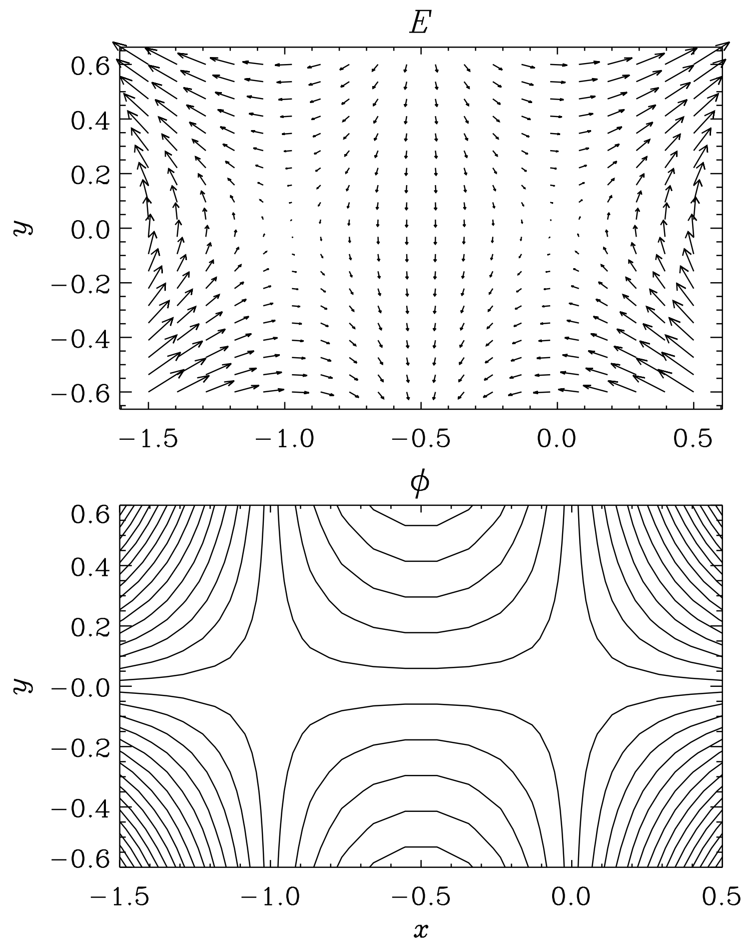

Figure 3: Electric field vectors and contours of constant electrostatic potential.

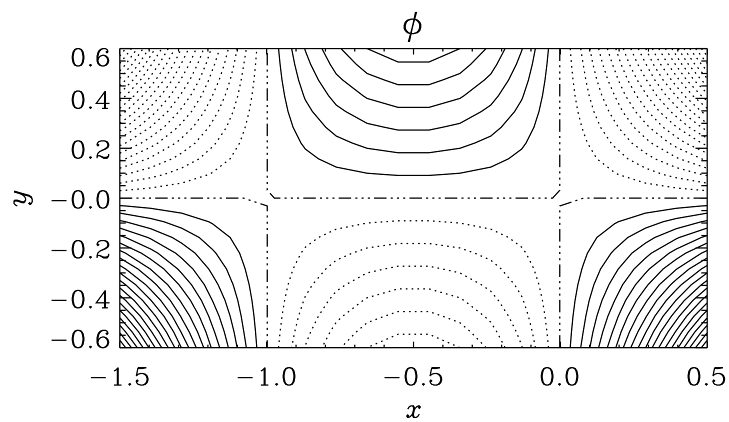

Figure 4: Electrostatic potential, where dotted lines denote negative values of ϕ. As expected **E** points in the direction in which ϕ becomes most negative (e.g. the upper left and right corners.

(a) Solve this equation for $I(x)$ at all points between $x = 0$ and $x = D$.

(b) Define the quantity $\tau = \alpha D$. Give an approximate solution for the "emergent intensity" $I(D)$ under the three limiting cases:

- $\tau \ll 1$.
- $\tau \gg 1$ and $S \gg I_0$.
- $\tau \gg 1$ and $S \ll I_0$.

(c) Each of the three above cases matches with one of the following three physical analogies. Which do you think corresponds to which, and why?

- Shining a flashlight through a piece of dark smoky quartz.
- Shining a flashlight through the bright flame of a welder's torch.
- Shining a flashlight through a glass window pane.

Hint: The quantity τ can be thought of as the "optical depth" or opaqueness of the slab—i.e., how efficiently does the gas absorb (or otherwise eliminate) the incoming beam. The quantity S is a "source function" that describes how the gas in the slab generates its own light.

. . . .

(a) We can sort of separate the variables to get

$$\frac{dI}{dx} + \alpha I = \alpha S,$$

but we can't quite integrate this equation yet unless we introduce a so-called integrating factor $e^{-\alpha x}$ and substitute

$$I(x) = e^{-\alpha x}\tilde{I}(x).$$

Inserting this gives

$$-\alpha I + e^{-\alpha x}\frac{d\tilde{I}}{dx} + \alpha I = \alpha S$$

where the αI cancels, so we have

$$e^{-\alpha x}\frac{d\tilde{I}}{dx} = \alpha S, \tag{1}$$

or

$$\frac{d\tilde{I}}{dx} = \alpha S e^{\alpha x}, \tag{2}$$

which can now be solved by separation of variables, i.e.

$$\tilde{I} - \tilde{I}_0 = \alpha S \int_0^x e^{\alpha x'}\, dx',$$

and because $\int e^{\alpha x}\, dx = \alpha^{-1}\, e^{\alpha x} + \text{const}$, we have

$$\tilde{I} = \tilde{I}_0 + S(e^{\alpha x} - 1).$$

We now express this solution in terms of I by substituting $\tilde{I} = e^{\alpha x} I$, so

$$e^{\alpha x} I = e^{\alpha \times 0} I_0 + S\,(e^{\alpha x} - 1),$$

where $e^{\alpha\times 0} I_0 = I_0$, and after multiplying by $e^{-\alpha x}$

$$I(x) = I_0 e^{-\alpha x} + S\,(1 - e^{-\alpha x}),$$

which is the solution! Several solutions are shown in Figs. 5 and 6.

As noted by some of you, in this particular case, since S = const, one can solve this equation also directly by separation of variables. Thus, we write

$$\frac{dI}{I-S} = -\alpha\, dx.$$

Since S = const, we can substitute dI by $d(I-S)$, but that means that the lower and upper boundary values are $I_0 - S$ and $I - S$, respectively, and the integrating variable will now be called $I' - S$. Thus, we have

$$\int_{I_0-S}^{I-S} \frac{d(I'-S)}{I'-S} = -\alpha \int_0^x dx'.$$

This gives

$$\ln(I-S) - \ln(I_0 - S) = -\alpha x,$$

or

$$\ln(I-S) = \ln(I_0 - S) - \alpha x.$$

Exponentiating this gives

$$I - S = (I_0 - S)\, e^{-\alpha x},$$

which can also be written in the form

$$I = I_0\, e^{-\alpha x} + S\,\left(1 - e^{-\alpha x}\right).$$

which is indeed the same as the solution via integrating factor.

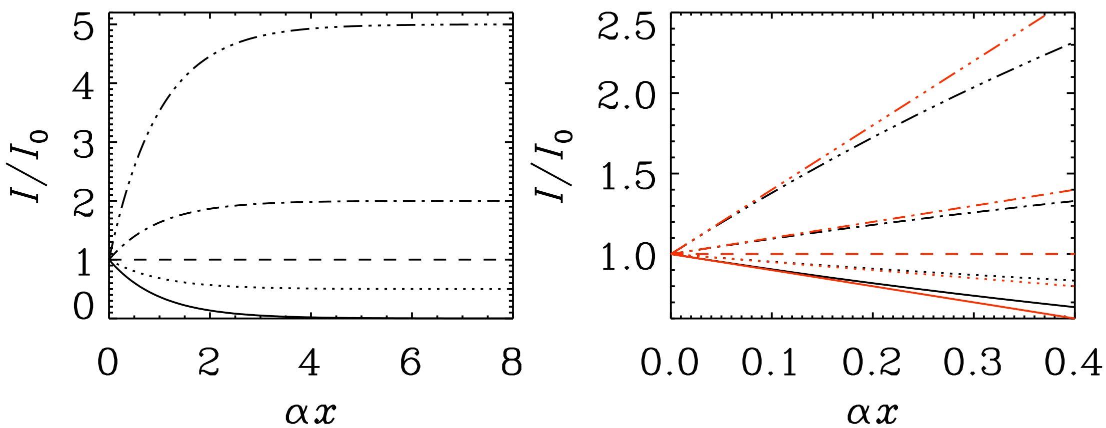

Figure 5: Dependence of $I(\alpha x)/I_0$ on αx for S/I_0=0 (solid), 0.5 (dotted), 1 (dashed), 2 (dash-dotted), 5 (dash-triple-dotted). The plot on the right shows a comparison with the approximation $\exp(x) \approx 1 - x$ (ed lines) over a shorter range $0 \le \alpha x \le 0.4$

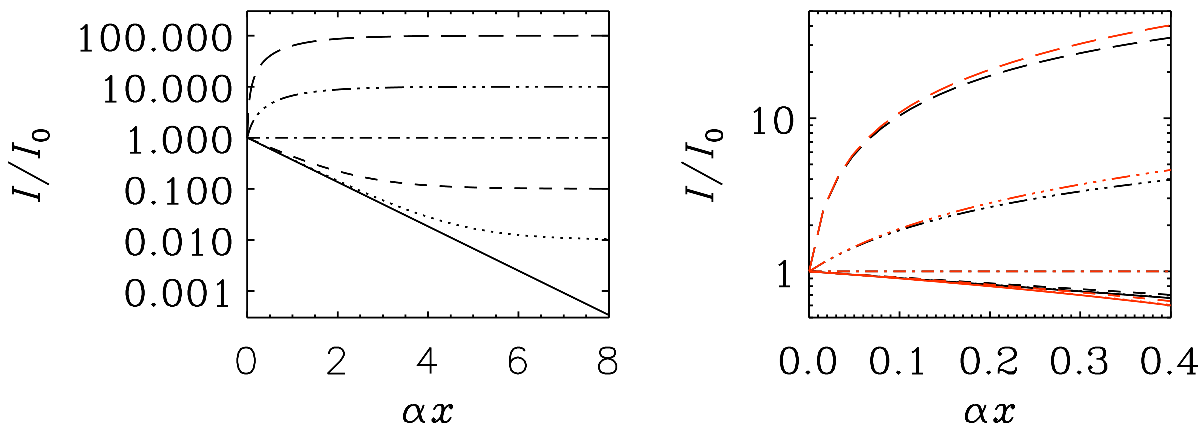

Figure 6: Similar to previous figure, but now in semi-logarithmic representation for S/I_0=0 (solid), 0.01 (dotted), 0.1 (dashed), 1 (dash-dotted), 10 (dash-triple-dotted), and 100 (long dashed).

(b) At $x = D$, the solution is given by

$$I(D) = I_0 e^{-\tau} + S\left(1 - e^{-\tau}\right) .$$

For $\tau \ll 1$, we can use the series expansion for the exponential function with a small argument,

$$e^{-\tau} \approx 1 - \tau + \cdots$$

(where we ignore tiny quantities like τ^2, τ^3, etc). Thus, in this limit,

$$I(D) \approx I_0(1-\tau) + \tau S$$

and if we took $\tau \to 0$, this leaves only

$$I(D) \approx I_0 .$$

For $\tau \gg 1$, the $e^{-\tau}$ term goes to zero extremely rapidly, and all that is left is

$$I(D) \approx S$$

and it *doesn't matter* whether $S \ll I_0$ or $S \gg I_0$, since in this limit the intensity at $x = D$ completely "forgets" about the boundary condition at $x = 0$.

(c) The glass window is the "optically thin" case of $\tau \ll 1$, since the intensity that comes out is approximately equal to the intensity I_0 that goes in.

The other two cases correspond to the "optically thick" limit of $\tau \gg 1$. The intensity that comes out is dominated solely by the source function S; i.e., by the properties of the slab itself. The dark smoky quartz absorbs the incoming beam but doesn't generate much light of its own; thus the emergent beam is less intense than what went in ($S \ll I_0$). For the welder's torch, the initial beam is swamped by the much brighter intrinsic emission from within the slab ($S \gg I_0$).

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4. Electromagnetic Energy Conservation. Use Maxwell's equations, for a vacuum environment (i.e., $\mathbf{D} = \epsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$), to show that

$$\frac{\partial}{\partial t}(U_E + U_B) + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}$$

where

$$U_{\mathrm{E}} = \frac{\epsilon_0 |\mathbf{E}|^2}{2} \ , \quad U_{\mathrm{B}} = \frac{|\mathbf{B}|^2}{2\mu_0} \ , \quad \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \ .$$

Hint: The online version of the "useful formulae" document contains a new vector identity that didn't get included in time for the printed handouts on the first day. If vectors **A** & **B** depend on time t, then the chain rule for the dot product is given by

$$\frac{\partial}{\partial t} (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} \ .$$

. . . .

There are several ways to go about it, but I think the most straightforward is to start by expanding out

$$\frac{\partial}{\partial t} (U_{\mathrm{E}} + U_{\mathrm{B}}) = \frac{\epsilon_0}{2} \frac{\partial}{\partial t} \left(|\mathbf{E}|^2 \right) + \frac{1}{2\mu_0} \frac{\partial}{\partial t} \left(|\mathbf{B}|^2 \right) = \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

which makes use of the vector identity above and the fact that $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$.

Solving Ampère's law and Faraday's law for the time derivatives of **E** and **B** allows us to substitute in to get

$$\frac{\partial}{\partial t} (U_{\mathrm{E}} + U_{\mathrm{B}}) = \frac{1}{\mu_0} [\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E})] - \mathbf{E} \cdot \mathbf{J} \ .$$

This is almost the final expression. The quantity in square brackets can be simplified, using a vector chain rule identity, to $\boldsymbol{\nabla} \cdot (\mathbf{B} \times \mathbf{E})$. We can switch the order of vectors in the cross product if we change the sign, so this quantity is also equal to $-\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B})$. Some algebra is all that is needed to rearrange everything into the desired form.
