ASTR-3760: Solar and Space Physics .... **Problem Set 2** (Due Mon., February 8, 2016)

Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

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1. Computing source function from observed center to limb variation. We use the equation of radiation transport to infer the depth dependence of the source function from the observed center to limb variation.

We use the gray approximation and write the equation of radiation transport in the form

$$\hat{\mathbf{n}} \cdot \nabla I = -\kappa\rho(I - S)$$

where $I(z, \hat{\mathbf{n}})$ is the intensity as a function of height z and direction $\hat{\mathbf{n}}$, κ is the opacity per unit mass, ρ is the density of the gas, and S is the source function.

(a) Derive the equation of radiation transport in the form

$$\mu \frac{dI}{d\tau} = I - S \tag{1}$$

where $\mu = \cos\theta$ is the cosine of the angle θ between the direction of the ray $\hat{\mathbf{n}}$ and the vertical direction, and τ is the optical depth. Thus, I is now a function of τ and μ.

(b) Verify that

$$I(\tau, \mu) = I(\tau_0, \mu)\, e^{-(\tau_0 - \tau)/\mu} + \int_{\tau}^{\tau_0} S(\tau')\, e^{-(\tau' - \tau)/\mu}\, d\tau'/\mu$$

obeys Eq. (1), where $I(\tau_0, \mu)$ is the intensity at some arbitrarily chosen reference value of the optical depth τ_0.

(c) Put $\tau = 0$ and $\tau_0 = \infty$, and show that

$$I(\mu) = \int_0^{\infty} S(\tau')\, e^{-\tau'/\mu}\, d\tau'/\mu$$

where $I(\mu)$ is the intensity at the position of the observer, who is located at $\tau = 0$, so $I(\mu)$ is the same as $I(0, \mu)$. This equation is an integral equation that can be inverted to obtain $S(\tau)$ for a given profile of $I(\mu)$.

(d) Drop the prime in the equation above and show that

$$\int_0^{\infty} \tau\, e^{-\tau/\mu}\, d\tau = \mu^2$$

and

$$\int_0^{\infty} \tau^2\, e^{-\tau/\mu}\, d\tau = 2\mu^3$$

(e) Insert for $S(\tau)$ the Taylor expansion

$$S(\tau) = S_0 + S_1\tau + \tfrac{1}{2}S_2\tau^2$$

where S_0, S_1, and S_2, are suitable coefficients and compute $I(\mu)$.

(f) Compute $I(\mu)$ for

$$S(\tau) = \frac{2}{5} + \frac{3}{5}\tau. \tag{2}$$

(g) Use the measured and tabulated values of $I(\mu)$ to compute $S(\tau)$.

μ	$I(\mu)$
1	1.0
2/3	0.8
1/3	0.5

(h) Describe how this new function $S(\tau)$ obtained from the tabulated values is different from that given by Eq. (2). Plot (or sketch) both $I(\mu)$ and $S(\tau)$.

(i) Instead of performing an expansion around $\tau = 0$ as under (e), expand around $\tau = \tau_* = 1$, using

$$S(\tau) = S_0 + S_1(\tau - \tau_*) + \tfrac{1}{2}S_2(\tau - \tau_*)^2, \tag{3}$$

with new coefficients S_0, S_1, and S_2, and repeat steps (f)–(h).

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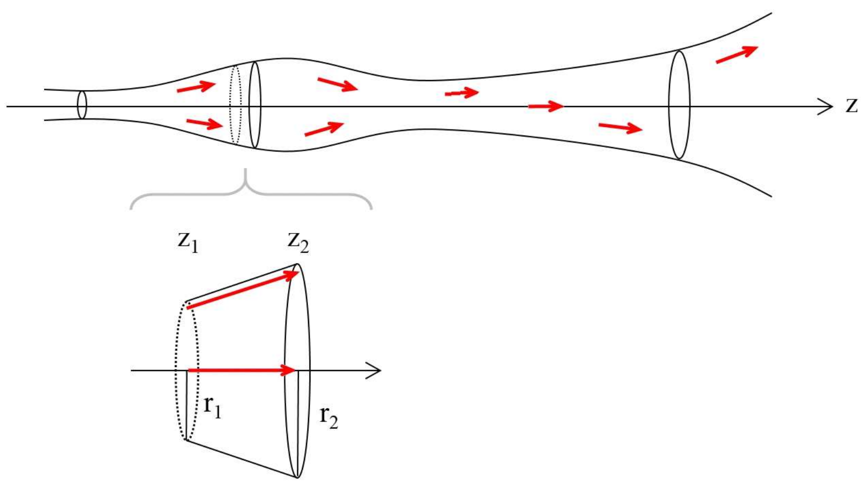
2. Thin magnetic flux tubes. We'd like to explore the real meaning of the most trivial-looking of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$.

Consider a bundle of magnetic field lines constrained to follow a sausage-like shape oriented mostly along the z axis:



We'll use cylindrical coordinates (r, ϕ, z) to describe spatial variations in $\mathbf{B}$, and assume axial symmetry around the z axis (i.e., no variations in ϕ, and $B_\phi = 0$).

(a) In the zoomed-in region between z_1 and z_2, assume that

$$\frac{\partial B_z}{\partial z} \quad \text{is a constant given by} \quad \frac{\Delta B_z}{\Delta z} = \frac{B_{z2} - B_{z1}}{z_2 - z_1}.$$

Integrate the $\nabla \cdot \mathbf{B} = 0$ equation to show that

$$B_r = C\,r$$

between z_1 and z_2, and solve for C in terms of the other variables of this problem. Describe any assumptions or boundary conditions that you had to use.

(b) Note that, along the outer edge of the tube, the vector **B** points along the direction of the tube's geometric outline. Considering only *small* relative changes in r between z_1 and z_2, use everything given so far to show that the change in cross-sectional area ($A = \pi r^2$) from z_1 to z_2 is given by

$$\frac{\Delta A}{A_1} \approx -\frac{\Delta B_z}{B_{z1}} .$$

Hint: The binomial expansion for the quantity $[1+(\Delta r/r_1)]^2$, where $\Delta r = (r_2 - r_1) \ll r_1$, might be useful to use at some point.

(c) Lastly, assume that all "Δ" changes are infinitesimally small, in comparison to the values of the quantities at z_1, and show that

$$A(z)\,B_z(z) = \text{ constant}$$

i.e., that *magnetic flux is conserved* along a thin flux tube.

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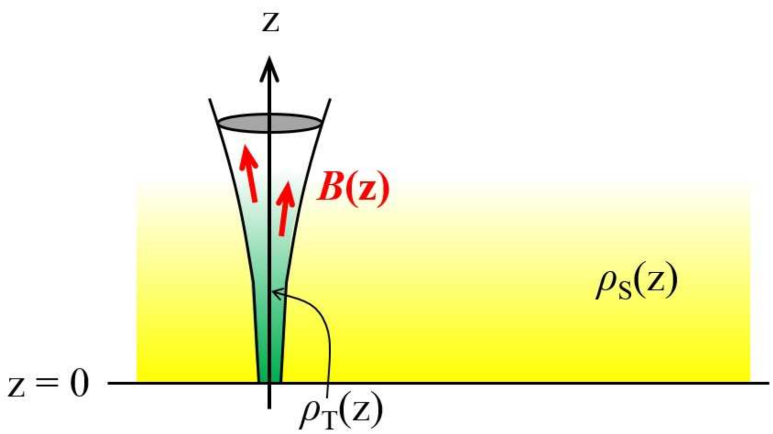
3. Magnetic flux tubes in the Sun's atmosphere. Consider a small piece of the solar surface, in which thin flux tubes with strong **B** are peppered through larger regions of weak/negligible **B**:



Let's assume the entire region shown above is isothermal (i.e., constant T throughout), and the z dependence of density ρ obeys the hydrostatic equilibrium derived in class,

$$\rho(z) = \rho_0 \exp\left(-\frac{z}{H}\right) \quad \text{where} \quad H = \frac{k_B T}{\mu m_H g} .$$

where the density at the base ($z = 0$) inside the tube $\rho_{0,T}$ does not necessarily equal the base density in the surroundings $\rho_{0,S}$.

(a) If the plasma inside the tube is in total pressure balance with the plasma outside the tube, show that the z dependence of magnetic field strength B inside the tube obeys

$$B(z) = B_0 \exp\left(-\frac{z}{K}\right)$$

and solve for B_0 and K in terms of the other properties of the system.

(b) Using the principle of magnetic flux conservation from the previous problem, note that if $B(z)$ decreases with increasing height, then the cross-sectional area of the tube $A(z)$ must increase. If the tubes occupy 1% of the solar surface at the lower boundary ($z = 0$), then at what height will they fill the entire surface? Solve for this "merging height" both in terms of the other variables and also as an actual number (in units of km) for the real solar case of $T \approx 5000$ K and $\mu \approx 1.3$.