Please try to be neat when writing up answers. In cases where calculations are called for, please show all of the intermediate steps, including any approximations you choose to make and any sketches you may need to illustrate what's what. Be careful to properly evaluate units and significant figures. Calculations given without 'showing the work' will receive zero credit, even if the final answer is correct.

1. Use dimensional arguments to determine the form of the energy spectrum $E(k)$ for hydromagnetic turbulence. You may assume that the spectrum can be written in the form

$$E(k) = C\,(v_A \epsilon)^a k^b,$$

where C is a dimensionless constant, v_A is the Alfvén speed, ϵ (with dimension $\mathrm{m^2\,s^{-3}}$) the energy injection rate, and k the wavenumber.

[Note that $\int E(k)dk$ has the dimension $\mathrm{m^2\,s^{-2}}$.]

2. Your favorite space weather story:
   (a) Describe in a few words your favorite space weather story.
   (b) Try to complement the story with some simple numerical estimates to check whether it makes sense.
   (c) How frequent is an event of the type you described?

3. Assume that the magnetic field, $\boldsymbol{B}$, is governed by the equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0, \quad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}/\mu_0, \quad \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B},$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ the current density, $\boldsymbol{v}$ the velocity, and μ_0 the permeability.

   (a) Show that

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} = 0.$$

   (b) Write $\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \epsilon_{ijk}\partial_i(E_j B_k)$ and show, using the product rule, that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}.$$

   [Remember that $\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{jik}$.]

   (c) Use this relation to show that

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) + \boldsymbol{J} \cdot \boldsymbol{E} = 0.$$

   (d) Show that the energy equation can be written in the form

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) + \boldsymbol{v} \cdot (\boldsymbol{J} \times \boldsymbol{B}) = 0.$$

4. Use the Eddington solution, $S(\tau) = (3/4\pi)F\,(\tau + \frac{2}{3})$, in the formal solution for radiative transfer for

   (a) upward propagating rays

   $$I(\tau,\mu) = e^{\tau/\mu} \int_{\tau}^{\infty} S(\tau')\, e^{-\tau'/\mu}\, d\tau'/\mu \quad \text{(for upward rays, } \mu > 0), \tag{1}$$

   to show that

   $$I(\tau,\mu) = (3/4\pi)\,F\,(\tau + \tfrac{2}{3} + \mu) \qquad\qquad \text{(for } \mu > 0), \tag{2}$$

   (b) downward propagating rays

   $$I(\tau,\mu) = e^{\tau/\mu} \int_{\tau}^{0} S(\tau')\, e^{-\tau'/\mu}\, d\tau'/\mu \quad \text{(for downward rays, } \mu < 0) \tag{3}$$

   to show that

   $$I(\tau,\mu) = (3/4\pi)F\,[(\tau + \tfrac{2}{3} + \mu) - (\tfrac{2}{3} + \mu)\, e^{\tau/\mu}] \quad \text{(for } \mu < 0). \tag{4}$$

5. Compute *numerically* for $\tau = 1/3$ the first three moments of the intensity

   $$J(\tau) = \tfrac{1}{2} \int_{-1}^{1} I(\tau,\mu)\, \mathrm{d}\mu \tag{5}$$

   $$H(\tau) = \tfrac{1}{2} \int_{-1}^{1} I(\tau,\mu)\, \mu\, \mathrm{d}\mu \tag{6}$$

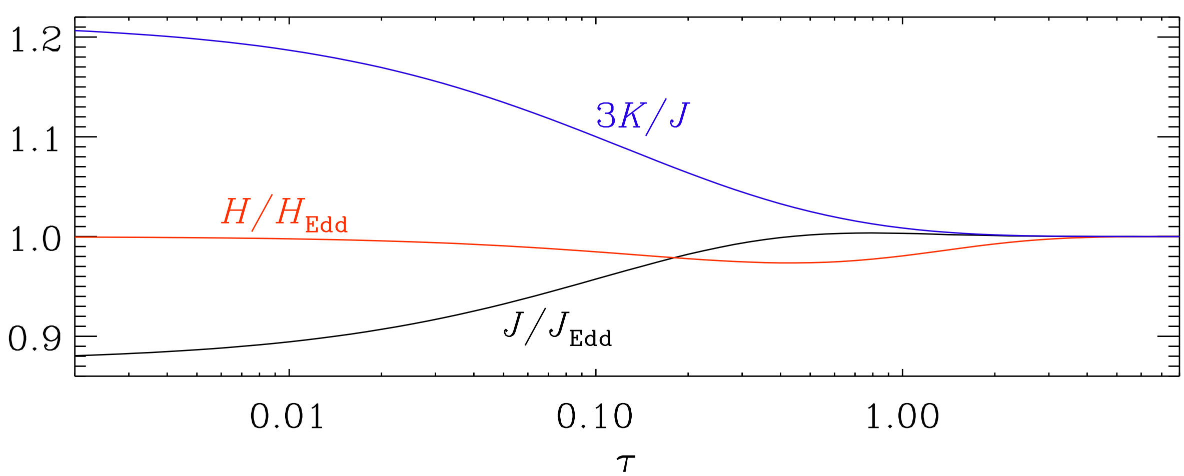
   $$K(\tau) = \tfrac{1}{2} \int_{-1}^{1} I(\tau,\mu)\, \mu^2\, \mathrm{d}\mu \tag{7}$$

   where

   $$I(\tau,\mu) = (3/4\pi)\,F\,(\tau + \tfrac{2}{3} + \mu) \qquad\qquad \text{(for } \mu > 0), \tag{8}$$

   $$I(\tau,\mu) = (3/4\pi)F\,[(\tau + \tfrac{2}{3} + \mu) - (\tfrac{2}{3} + \mu)\, e^{\tau/\mu}] \quad \text{(for } \mu < 0). \tag{9}$$

   Hint: the result should agree with the figure below at $\tau = 1/3$.



6. For a steady spherically symmetric flow of an isothermal gas with constant sound speed $c_{\rm s}$, the Euler equation with a suitable body force is

$$u_r \frac{\mathrm{d}u_r}{\mathrm{d}r} = -c_{\rm s}^2 \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}. \tag{1}$$

The continuity equation is

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 \rho u_r) = 0. \tag{2}$$

(a) Show that the critical radius r_* where $|u_r| = c_{\rm s}$, is $r_* = GM/(2c_{\rm s}^2)$.

(b) Integrate Eqs (1) and (2), and then eliminate $\ln\rho$ to show that

$$\tfrac{1}{2}u_r^2 - c_{\rm s}^2 \ln|u_r| - 2c_{\rm s}^2 \ln r - \frac{GM}{r} = \tfrac{1}{2}c_{\rm s}^2 - c_{\rm s}^2 \ln c_{\rm s} - 2c_{\rm s}^2 \ln r_* - \frac{GM}{r_*}. \tag{3}$$

(c) Show that Eq. (3) can be written as

$$\mathcal{M} = \sqrt{C + 2\ln\mathcal{M}}, \tag{4}$$

where $\mathcal{M} = |u_r|/c_{\rm s}$ and

$$C = 4\left(\ln\tilde{r} + \frac{1}{\tilde{r}}\right) - 3,$$

with $\tilde{r} = r/r_*$.

(d) Calculate the value of C for $\tilde{r} = 10$, and find the corresponding value of $\mathcal{M}$ using three iteration steps starting with $\mathcal{M} = 1$. Show your working in all intermediate steps. Sketch the solution for $\mathcal{M}$ against $\tilde{r}$, and indicate the points where $\tilde{r} = 1$ and 10.

(e) Show that Eq. (4) can also be written as $\mathcal{M} = \exp\left[\frac{1}{2}(\mathcal{M}^2 - C)\right]$, and, for the same value of C, iterate for $\mathcal{M}$ starting again with $\mathcal{M} = 1$ (use three iterations, show your working). Again, sketch the solution of $\mathcal{M}$ against $\tilde{r}$, indicate the points where $\tilde{r} = 1$ and 10, and show the direction of the flow. In what area of stellar physics can this model be applied?

7. Consider a one-dimensional shock. Use the ideal fluid equations in conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0,$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e) + \frac{\partial}{\partial x}\left[v\left(\tfrac{1}{2}\rho v^2 + \rho e + p\right)\right] = 0,$$

where e is the internal energy density per unit mass, and the other variables have their usual meaning. Assume a perfect gas with

$$p = (\gamma - 1)\rho e.$$

(a) Why is it useful to consider a frame of reference comoving with the shock? Show that in a frame comoving with the shock the following three quantities are conserved:

$$J = \rho v; \tag{1}$$

$$I = \rho v^2 + p; \tag{2}$$

$$E = \tfrac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}. \tag{3}$$

(b) Eliminate first p/ρ and then ρ, to show that

$$\frac{v_2}{v_1} = \frac{2\gamma}{\gamma + 1}\left(1 + \frac{p_1}{\rho_1 v_1^2}\right) - 1$$

where the subscripts 1 and 2 refer, respectively, to the upstream and downstream sides of the shock.

(c) Calculate v_2 for $v_1 = 5$, $\rho_1 = p_1 = 1$ and $\gamma = 5/3$.

(d) Calculate ρ_2 and p_2. Sketch the velocity and density profiles indicating the positions of the upstream and downstream sides.

(e) State whether the normalised entropy,

$$s = \frac{1}{\gamma}\ln p - \ln \rho$$

is increased or decreased behind the shock. Calculate s_1 and s_2.