1. Stable & unstable stratification of an atmosphere.

(a) Explain qualitatively when the stratification of an atmosphere is stable to convection (use a sketch of specific entropy versus height).

(b) There is a critical temperature gradient, $\beta_{\rm crit} = (dT/dz)_{\rm crit}$, above which the stratification becomes unstable to convection. Show that

$$\beta_{\rm crit} = -\left(1 - \frac{1}{\gamma}\right) g \frac{\mu}{\mathcal{R}},$$

where g is gravity, γ the ratio of specific heats, $\mathcal{R}$ the universal gas constant and μ the mean molecular weight. [Hints: use the condition $c_p^{-1} ds/dz = 0 = \gamma^{-1} d\ln p/dz - d\ln\rho/dz$ for adiabatic stratification, write this in terms of p and T using the perfect gas equation $p = (\mathcal{R}/\mu)\rho T$, and eliminate p using the equation of hydrostatic equilibrium, $dp/dz = -\rho g$.]

(c) Consider an isothermal model of the upper layers of the Sun. Estimate the scale height $H = c_s^2/g$, using $c_s = 7\,{\rm km\,s^{-1}}$, and $g = GM/R^2$ (you may take $G = 7\times10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, $M = 2\times10^{30}\,{\rm kg}$, and $R = 7\times10^{8}\,{\rm m}$).

(d) Calculate the entropy gradient

$$\frac{1}{c_p}\frac{ds}{dz} = \left(1 - \frac{1}{\gamma}\right)\frac{1}{H}$$

with $\gamma = 5/3$.

(e) A hot bubble is in pressure equilibrium, but with a 10% density deficit relative to the surroundings. Calculate how far it will rise before reaching equilibrium.

(f) What are buoyancy oscillations (also known as Brunt-Väisälä oscillations)? What is the relevant restoring force? Give a mathematical expression of the force per unit mass, i.e., the acceleration.

(g) Estimate (within a factor of 3) the period $T_{\rm BV} = 2\pi/\omega_{\rm BV}$, where

$$\omega_{\rm BV} = \left(1 - \frac{1}{\gamma}\right)^{1/2}\frac{g}{c_s}, \tag{1}$$

for the solar atmosphere (radius from the center $r = 700\,{\rm Mm}$, sound speed $c_{\rm s} = 6\,{\rm km\,s^{-1}}$) and the bottom of the solar convection zone ($r = 500\,{\rm Mm}$, $c_{\rm s} = 200\,{\rm km\,s^{-1}}$). Use $\gamma = 5/3$, $g = GM/R^2$, where $G = 7\times10^{-11}\,{\rm m^3\,kg^{-1}\,s^{-2}}$, and $M = 2\times10^{30}\,{\rm kg}$. Give the period in seconds, minutes, or hours, as appropriate.

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(a) An initially buoyant blow has lower density than the environment and therefore an entropy excess relative to the environment ($\delta S/c_{\rm p} = -\delta\ln\rho$). When the environment has a positive gradient of specific entropy (see Fig. 1), the blob will rise until its entropy matches that of the environment.

It will be buoyantly neutral at this point. But because of the blob's inertial, it will overshoot by a certain amount. It will then have an entropy deficit relative to the environment, so it will be heavier and eventually fall back, so it is stable and there is no run-away (as in the unstable case).

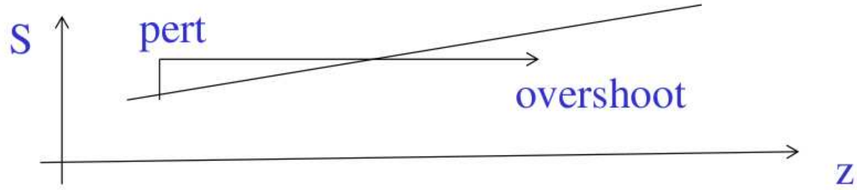

Figure 1: Positive gradient of specific entropy (see Lecture 15).

(b) Since the entropy s is constant we have

$$0 = \frac{1}{\gamma}\frac{d\ln p}{dz} - \frac{d\ln\varrho}{dz} = \frac{1}{\gamma}\frac{d\ln p}{dz} - \left(\frac{d\ln p}{dz} - \frac{d\ln T}{dz}\right) = \frac{d\ln T}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{d\ln p}{dz}\,,$$

because $\varrho \propto p/T$. So we have

$$\frac{d\ln T}{dz} = \left(1-\frac{1}{\gamma}\right)\frac{d\ln p}{dz}\,. \tag{2}$$

Using hydrostatic equilibrium, we have

$$\frac{d\ln p}{dz} = \frac{1}{p}\frac{dp}{dz} = -\frac{\varrho}{p}g = -\frac{g}{\mathcal{R}T/\mu}\,. \tag{3}$$

Combining Eqs. (2) and (3), we have

$$\frac{dT}{dz} = -\left(1-\frac{1}{\gamma}\right)g\frac{\mu}{\mathcal{R}}\,.$$

Since we did this calculation assuming $s=$ const (adiabaticity) this is then indeed the *critical* temperature gradient, $\beta_{\rm crit}$.

(c) Compute $g = GM/R^2 \approx 290\,{\rm m\,s^{-2}}$ with the numbers provided. Thus, $H = c_{\rm s}^2/g = 170\,{\rm km}$.

(d) For $\gamma = 5/3$, we have $1-1/\gamma = 2/5 = 0.4$, so the entropy gradient is $0.4/1.7\times10^5\,{\rm m^{-1}} = 2.4\times10^{-6}\,{\rm m^{-1}}$.

(e) A density deficit of 10% means that $\delta\ln\rho = -0.1$ and therefore we have $\delta s/c_p = 0.1$. To get the height, we need to divide this by the slope of the curve, i.e.,

$$\text{height} = 0.1/2.4\times10^{-6}\,{\rm m^{-1}} = 42{,}500\,{\rm m} \approx 40\,{\rm km}.$$

(f) Brunt-Väisälä oscillations are vertical up & down motions in a stably stratified atmosphere. The relevant restoring force is gravity or the buoyancy force, $\delta\rho\,g$.

(g) Inserting $g = GM/R^2$, the expression for the buoyancy or Brunt-Väisälä frequency is

$$\omega_{\rm BV} = \left(1-\frac{1}{\gamma}\right)^{1/2}\frac{GM}{R^2 c_{\rm s}}, \tag{4}$$

With $1-1/\gamma = 0.4$, and hence $\sqrt{1-1/\gamma}\approx 0.6$, we have

$$\omega_{\rm BV} = 0.6\,\frac{7\times10^{-11}\cdot 2\times10^{30}}{(0.7\times10^9)^2\cdot 0.6\times10^4}, \tag{5}$$

which gives $\omega_{\rm BV} \approx 0.03\,{\rm s}^{-1}$ and for the period $2\pi\omega_{\rm BV} \approx 4\,{\rm min}$ at the surface, and

$$\omega_{\rm BV} = 0.6\,\frac{7\times10^{-11}\cdot 2\times10^{30}}{(0.5\times10^{9})^2\cdot 2\times10^{5}}, \tag{6}$$

which gives $\omega_{\rm BV} \approx 1.7\times10^{-3}\,{\rm s}^{-1}$ and for the period $2\pi\omega_{\rm BV} \approx 60\,{\rm min}$ at the bottom of the convection zone, i.e., one hour.

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2. Momentum and energy equations in conservative forms. Consider the continuity, momentum, and energy equations in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{7}$$

and

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}p = 0, \tag{8}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{9}$$

where e is the internal energy per unit mass (which was called u in the lecture).

(a) Derive the evolution equation for the momentum density

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij}p) \tag{10}$$

Note that summation over double indices is assumed!

(b) Explain why this equation is in *conservative* form. Discuss how the volume-integrated momentum changes for periodic boundary conditions. What other boundary conditions give the same result?

(c) Derive the so-called total energy equation in the form

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j\left(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e + p\right)\right], \tag{11}$$

Again, summation over double indices is assumed.

(d) Explain in words how these equations can be used to say something about hydrodynamic planar shocks, where density, pressure, and density can discontinuously across a surface. Consider a one-dimensional frame of reference *comoving* with the shock. What happens to the time derivative in that frame? Use the equation of state in the form

$$p = (\gamma-1)\rho e$$

and count how many unknowns do you have?

(a) Using the product rule, we have

$$\frac{\partial \rho u_i}{\partial t} = \rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} \tag{12}$$

Likewise

$$\frac{\partial}{\partial x_j}[(\rho u_j)u_i] = (\rho u_j)\frac{\partial}{\partial x_j}u_i + u_i\frac{\partial}{\partial x_j}(\rho u_j) \tag{13}$$

With this we can write

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_j u_i)=\rho\frac{\mathrm{D}u_i}{\mathrm{D}t}+u_i\left(\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right). \tag{14}$$

Inserting $\rho\mathrm{D}u_i/\mathrm{D}t=-\partial p/\partial x_i$, and noting that we can write $\partial p/\partial x_i=\partial(\delta_{ij}p)/\partial x_j$, we can pull this term underneath the $\partial/\partial x_j$ derivative and have

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_j u_i+\delta_{ij}p)=0 \tag{15}$$

(b) It is in conservative form, because the rate of change of the momentum $\rho\boldsymbol{u}$ is given by a negative divergence term. Integrating over a certain volume, the rate of change of the integrated momentum is given by the surface integral of the momentum tensor, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho u_i\,\mathrm{d}V=-\oint(\rho u_j u_i+\delta_{ij}p)\,\mathrm{d}S_j. \tag{16}$$

In vanishes for periodic boundary conditions, for example, but it also vanishes of ρ and p vanish on the boundary.

(c) We use the energy and momentum equations,

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t}=-p\boldsymbol{\nabla}\cdot\boldsymbol{u}-\boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{17}$$

as well as the momentum equation,

$$\rho u_i\frac{\mathrm{D}u_i}{\mathrm{D}t}=-\boldsymbol{u}\cdot\boldsymbol{\nabla}p \tag{18}$$

Add the two gives

$$\frac{\partial}{\partial t}(\rho e+\tfrac{1}{2}\rho\boldsymbol{u}^2)+\frac{\partial}{\partial x_j}(\rho u_j e+\tfrac{1}{2}\rho u_j\boldsymbol{u}^2)=-\boldsymbol{\nabla}\cdot(p\boldsymbol{u}) \tag{19}$$

Thus, we have

$$\frac{\partial}{\partial t}(\rho e+\tfrac{1}{2}\rho\boldsymbol{u}^2)+\frac{\partial}{\partial x_j}(\rho u_j e+\tfrac{1}{2}\rho u_j\boldsymbol{u}^2+pu_j)=0 \tag{20}$$

or

$$\frac{\partial}{\partial t}(\rho e+\tfrac{1}{2}\rho\boldsymbol{u}^2)+\frac{\partial}{\partial x_j}\left[u_j(\rho e+\tfrac{1}{2}\rho\boldsymbol{u}^2+p)\right]=0 \tag{21}$$

(d) In a frame moving with the shock, the shock is stationary and therefore all time derivatives vanish, and therefore the divergences vanish. Since the shock is planar, we can assume it to move along the x direction, so the divergences are just x derivatives, and thus the terms underneath these x derivatives must be constant, i.e., equal when evaluated on both sides of the shock. Thus, we have

$$\rho u_x=\text{const}$$

$$\rho u_x^2+p=\text{const}$$

$$u_x(\tfrac{1}{2}\rho u_x^2+\rho e+p)=\text{const}$$

Together with the equation of state, we have4 equations for 4 unknowns, Assuming that we know the state of the shock on one side, then, we can use these 4 equations to solve for the 4 unknowns u_x, ρ, p, and e on the other side of the shock.

Incidently, the last of the three equations can be rewritten as $\rho u_x(\frac{1}{2}u_x^2 + e + p/\rho) = \text{const}$ so by using the first equation, the last one can be written as

$$\tfrac{1}{2}u_x^2 + e + p/\rho = \text{const}$$

which is of now a different constant.

3. Sound waves in a stratified atmosphere. Consider the continuity and momentum equations for an isothermal atmosphere with constant speed of sound, c_s, and uniform gravity, g, in one dimension,

$$\frac{\partial \rho}{\partial t} + u_z\frac{\partial \rho}{\partial z} + \rho\frac{\partial u_z}{\partial z} = 0, \tag{22}$$

$$\rho\frac{\partial u_z}{\partial t} + \rho u_z\frac{\partial u_z}{\partial z} + c_s^2\frac{\partial \rho}{\partial z} + \rho g = 0, \tag{23}$$

where ρ is density and u_z vertical velocity. Instead of using subscripts 0 and 1 for equilibrium and perturbed solutions, as we did in the lecture, we use here overbars and primes instead. Here, the quantities with an overbar are not necessarily constant. By contrast, the subscript 0 refers now to a constant coefficient.

(a) Show that the solution for hydrostatic equilibrium, $u_z = \overline{u_z} = 0$, is $\rho = \overline{\rho}(z) = \rho_0\, e^{-z/H}$, where ρ_0 is a constant and $H = c_s^2/g$ is the vertical scale height.

(b) Write $\rho = \overline{\rho} + \rho'$ and $u_z = u_z'$ and linearize equations (27) and (28) with respect to ρ' and u_z'.

(c) Assume that ρ' and $u_z = u_z'$ take the form

$$\rho'(z,t) = \rho_1 e^{ikz - i\omega t - z/2H}, \tag{24}$$

$$u_z'(z,t) = w_1 e^{ikz - i\omega t + z/2H}, \tag{25}$$

and show that the linearized equations can be written as

$$\begin{pmatrix} -i\omega & [ik - (2H)^{-1}] \\ [ik + (2H)^{-1}]c_s^2 & -i\omega \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_0 w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

(d) Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.

(e) Give a qualitative plot of the dispersion relation.

(f) Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6\,\mathrm{km/s}$ and $g = 270\,\mathrm{ms}^2$.

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(a) In hydrostatic equilibrium we have

$$c_\mathrm{s}^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} = -g$$

so $\ln\rho/\rho_0 = -gz/c_\mathrm{s}^2$ and therefore $\rho = \rho_0\exp(-gz/c_\mathrm{s}^2)$, which we write as $\rho = \rho_0\exp(-z/H)$, where $H = c_\mathrm{s}^2/g$ is the scale height.

(b) The linearized equations take the form

$$\frac{\partial \rho'}{\partial t} + u_z'\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} + \overline{\rho}\frac{\partial u_z'}{\partial z} = 0, \tag{27}$$

$$\overline{\rho}\frac{\partial u_z'}{\partial t} + c_s^2\frac{\partial \rho'}{\partial z} + \rho' g = 0, \tag{28}$$

(c) Inserting Eqs. (24) and (25), we have

$$-i\omega\rho_1 e^{ikz-i\omega t-z/2H} + w_1 e^{ikz-i\omega t+z/2H}\left(-\frac{\rho_0}{H}\,e^{-z/H}\right) + \rho_0 e^{-z/H}\left(ik+\frac{1}{2H}\right) w_1 e^{ikz-i\omega t+z/2H} = 0 \tag{29}$$

$$-i\omega\rho_0 e^{-z/H} w_1 e^{ikz-i\omega t+z/2H} + c_s^2\left(ik-\frac{1}{2H}\right)\rho_1 e^{ikz-i\omega t-z/2H} + g\rho_1 e^{ikz-i\omega t-z/2H} = 0, \tag{30}$$

Note that in both equations the exponential factors cancel, which requires in some expressions the presence of the $e^{-z/H}$ factors from the background density. Thus, we have

$$-i\omega\rho_1 + w_1\left(-\frac{\rho_0}{H}\right) + \rho_0\left(ik+\frac{1}{2H}\right) w_1 = 0 \tag{31}$$

$$-i\omega\rho_0 w_1 + c_s^2\left(ik-\frac{1}{2H}\right)\rho_1 + g\rho_1 = 0, \tag{32}$$

using $g = c_\mathrm{s}^2/H$, and combining terms, we have

$$-i\omega\rho_1 + \left(ik-\frac{1}{2H}\right)\rho_0 w_1 = 0 \tag{33}$$

$$-i\omega\rho_0 w_1 + c_s^2\left(ik+\frac{1}{2H}\right)\rho_1 = 0 \tag{34}$$

In matrix form, this can be written as

$$\begin{pmatrix} -i\omega & [ik-(2H)^{-1}] \\ [ik+(2H)^{-1}]c_s^2 & -i\omega \end{pmatrix}\begin{pmatrix}\rho_1 \\ \rho_0 w_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{35}$$

(d) The determinant of the matrix vanishes when

$$-\omega^2 - \left(-k^2-\frac{1}{4H^2}\right)c_\mathrm{s}^2 = 0 \tag{36}$$

or

$$\omega^2 = c_\mathrm{s}^2 k^2 + \omega_0^2 \tag{37}$$

(e) Fig. 2 shows two graphic representations of the dispersion relation.

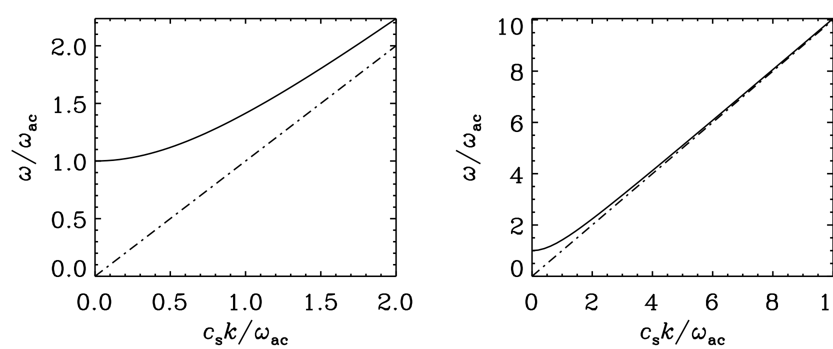

Figure 2: Dispersion relation.

(f) Inserting the numerical values, we have $\omega_0 = c_\mathrm{s}/2H = g/2c_\mathrm{s} = 270/12,000\,\mathrm{s}^{-1} = 0.0225\,\mathrm{s}^{-1}$, so the period is $280\,\mathrm{s} = 4.7\,\mathrm{min}$.