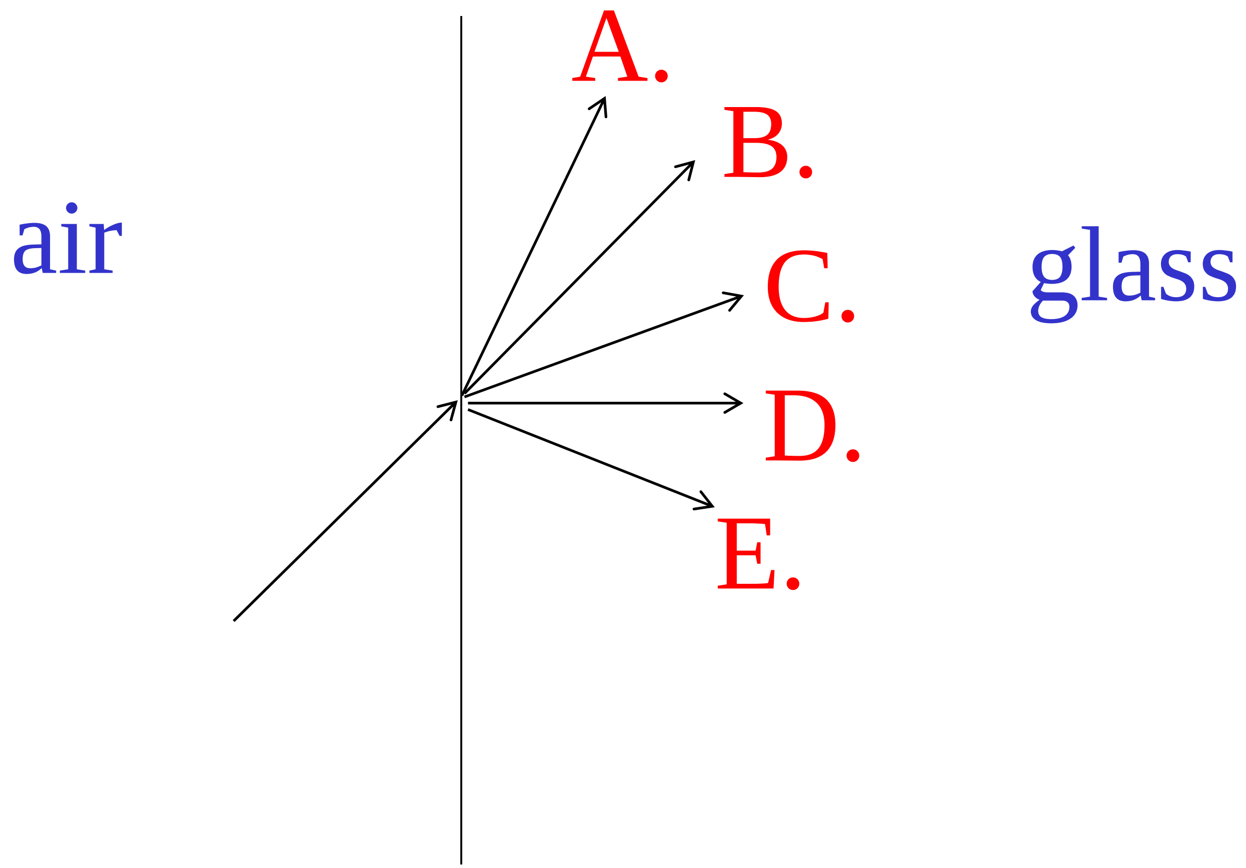
Lecture 10

- Newton’s law
- Equation of state
- Lorentz force

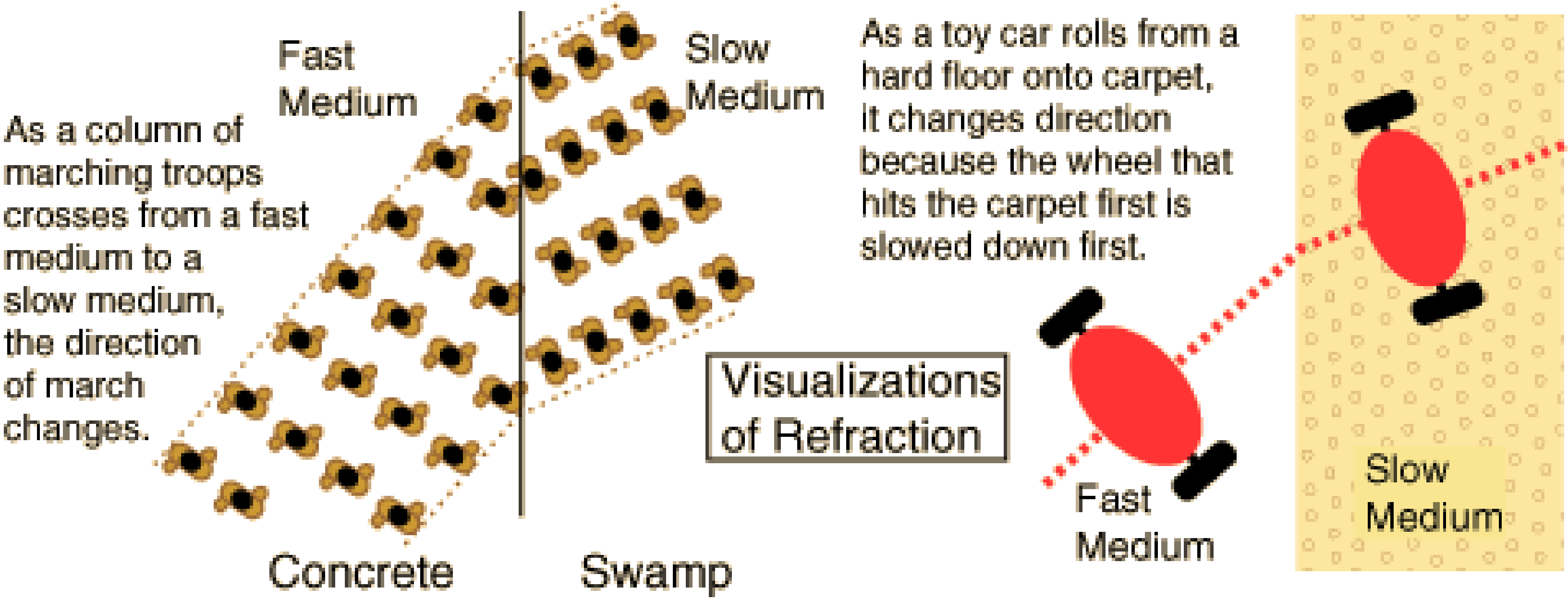
Summary of previous lecture

- Refraction
- Wave equation (light waves)

Which angle does it bend?



Refraction analogy



Newton's law → Momentum equation

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F} + m\mathbf{g} + \ldots$$

per unit volume

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{f} + \rho\mathbf{g} = \ldots$$

sum of all forces per unit volume

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g}\ldots$$

Mean molecular weight

Pressure $p = nk_B T$ Depends on # of particles

Neutral hydrogen $\rho = nm_{\mathrm{H}}$ Gas density

$$p = \rho \frac{k_B}{m_{\mathrm{H}}} T = \rho \frac{\Re}{\mu} T$$

Universal gas constant

$$\frac{k_B}{m_{\mathrm{H}}} = \Re = \frac{1.38\times10^{-23}\ \mathrm{kg\ m^2\ s^{-2}\ K^{-1}}}{1.66\times10^{-27}\ \mathrm{kg}} = 8315\ \mathrm{m^2\ s^{-2}\ K^{-1}}$$

Ionized hydrogen or neutral helium

Increased pressure $p = 2n_{\mathrm{H}}k_{\mathrm{B}}T$ 2 particles per H atom

Ionized hydrogen $\rho = \frac{n}{2}m_{\mathrm{H}}$ Gas density

$$p = \rho\frac{k_B}{m_{\mathrm{H}}/2}T = \rho\frac{\Re}{\mu}T$$

so $\mu = 1/2$

Neutral helium $\rho = 4nm_{\mathrm{H}}$

so $\mu = 4$

Hydrogen-helium mixture, neutral

Fractional number densities

$$n_{\mathrm{H}} = \frac{\rho X}{m_{\mathrm{H}}} \qquad n_{\mathrm{He}} = \frac{\rho Y}{4 m_{\mathrm{H}}}$$

so

$$n = \frac{\rho}{m_{\mathrm{H}}}\left(X + \frac{Y}{4} \right)$$

and thus

$$\mu = \frac{1}{X + Y/4} = \frac{1}{0.7 + \frac{0.3}{4}} = \frac{1}{\frac{28}{40} + \frac{3}{40}} = \frac{40}{31} = 1.3$$

Fully ionized H+He mixture

A. $\mu = \frac{1}{X + Y/4}$

B. $\mu = \frac{2}{X + Y/4}$

C. $\mu = \frac{1}{2X + Y/2}$

D. $\mu = \frac{1}{2X + Y}$

Stratification

sum of all forces per unit volume

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} \ldots$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left(\frac{\Re T}{\mu} \rho \right) + \rho \mathbf{g} \ldots$$

Isothermal case (T=const)

$$\rho \frac{d\mathbf{u}}{dt} = -\frac{\Re T}{\mu} \nabla \rho + \rho \mathbf{g} \ldots$$

Isothermal stratification

Hydrostatic case

$$0 = -\frac{\Re T}{\mu}\nabla\rho + \rho\mathbf{g}$$

Cartesian coordinates
gravity downward

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}, \quad g > 0$$

3 equations

$$\frac{\partial\rho}{\partial x} = 0, \qquad \frac{\partial\rho}{\partial y} = 0, \qquad \frac{\Re T}{\mu}\frac{\partial\rho}{\partial z} = -\rho g$$

Integrate

z component

$$\frac{\Re T}{\mu}\frac{d\rho}{dz} = -\rho g$$

only z dependent

$$\rho = \rho(z)$$

Separation of variables

$$\frac{d\rho}{\rho} = -\frac{g}{\frac{\Re T}{\mu}}dz$$

define scale height

$$H = \frac{\Re T}{\mu g}$$

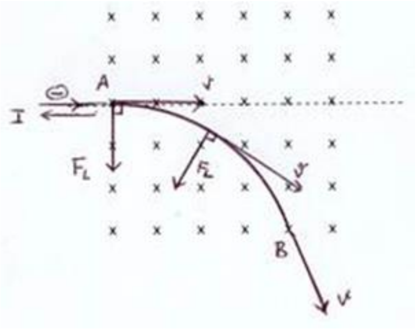
Lower & upper boundaries

$$\int_{\rho_0}^{\rho} d\ln\rho' = -\int_0^z \frac{dz'}{H}$$

Exponential stratification

$$\rho = \rho_0 \exp\left(-z/H\right)$$

Lorentz force



$$m\frac{d\mathbf{u}}{dt} = q\mathbf{u}\times\mathbf{B}$$

per unit volume, for hydrogen (H) and electrons (e)

$$\underbrace{nm}_{\rho}\frac{d\left(\mathbf{u}_{\mathrm{H}}+\mathbf{u}_{\mathrm{e}}\right)}{dt} = \underbrace{n\left(e\mathbf{u}_{\mathrm{H}}-e\mathbf{u}_{\mathrm{e}}\right)}_{\mathbf{J}}\times\mathbf{B}+\ldots$$

sum of all forces per unit volume

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p+\rho\mathbf{g}+\mathbf{J}\times\mathbf{B}+\ldots$$

Lorentz force (cont'd)

$$(\mathbf{J}\times\mathbf{B})_i = \left(\tfrac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right)_i = ?$$

use

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

apply

$$(\mathbf{B}\times(\nabla\times\mathbf{B}))_i = \varepsilon_{ijk}B_j\varepsilon_{klm}\partial_l B_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})B_j\partial_l B_m$$

$$= B_j\partial_i B_j - B_j\partial_j B_i$$

$$= \partial_i\left(\tfrac{1}{2}\mathbf{B}^2\right) - (\mathbf{B}\cdot\nabla)B_i$$

Magnetic pressure & tension forces

sum of all forces per unit volume

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g} + \ldots$$

sum of all forces per unit volume

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p \underbrace{- \frac{1}{2\mu_0} \nabla \mathbf{B}^2 + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}}_{= \mathbf{J} \times \mathbf{B}} + \rho \mathbf{g} + \ldots$$

combine

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left(p + \underbrace{\mathbf{B}^2 / 2\mu_0}_{p_{\mathrm{mag}}} \right) + \underbrace{\frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}}_{\text{tension force}} + \rho \mathbf{g} + \ldots$$

Tension force points…

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p \underbrace{- \frac{1}{\mu_0} \nabla \mathbf{B}^2 + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}}_{= \mathbf{J} \times \mathbf{B}} + \rho \mathbf{g} + \ldots$$

A. Along ***B***

B. Perpendicular to ***B***

C. Perpendicular to ***J***

D. Neither of them