Summary of previous lecture

- Newton's law
- Equation of state
- Lorentz force

Lecture 11

- Lorentz force (cont'd)
- Sound waves
- Flux tubes (div$\mathbf{B}$=0)

How we got the scale height

sum of all forces per unit volume

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} \ldots$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left(\frac{\Re T}{\mu} \rho \right) + \rho \mathbf{g} \ldots$$

Isothermal case

$$\rho \frac{d\mathbf{u}}{dt} = -\frac{\Re T}{\mu} \nabla \rho + \rho \mathbf{g} \ldots$$

Exponential stratification (isothermal)

z component

$$\frac{\Re T}{\mu}\frac{d\rho}{dz} = -\rho g$$

only z dependent

$$\rho = \rho(z)$$

Separation of variables

$$\frac{d\rho}{\rho} = -\frac{g}{\frac{\Re T}{\mu}} dz$$

define scale height

$$H = \frac{\Re T}{\mu g}$$

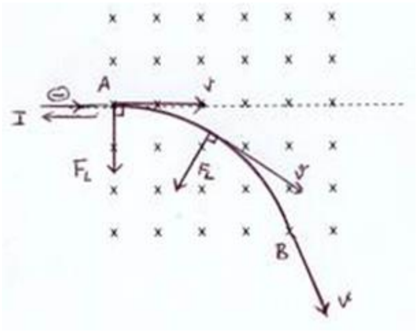
Lower & upper boundaries

$$\int_{\rho_0}^{\rho} d\ln\rho' = -\int_0^z \frac{dz'}{H}$$

Exponential stratification

$$\rho = \rho_0 \exp\left(-z/H\right)$$

Lorentz force: charge neutral



$$m\frac{d\mathbf{u}}{dt} = q\mathbf{u}\times\mathbf{B}$$

per unit volume, for hydrogen (H) and electrons (e)

$$\underbrace{nm}_{\rho}\frac{d\left(\mathbf{u}_{\mathrm{H}}+\mathbf{u}_{\mathrm{e}}\right)}{dt} = \underbrace{n\left(e\mathbf{u}_{\mathrm{H}}-e\mathbf{u}_{\mathrm{e}}\right)}_{\mathbf{J}}\times\mathbf{B}+\ldots$$

sum of all forces per unit volume

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p+\rho\mathbf{g}+\mathbf{J}\times\mathbf{B}+\ldots$$

In which direction…

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p \underbrace{- \frac{1}{\mu_0} \nabla \mathbf{B}^2 + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}}_{=\mathbf{J}\times\mathbf{B}} + \rho \mathbf{g} + \ldots$$

does $\mathbf{J} \times \mathbf{B}$ point?

A. Along ***B***
B. Perpendicular to ***B***
C. Perpendicular to ***J***
D. Neither of these
E. Perp to both ***J*** and ***B***

In which direction…

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p \underbrace{- \tfrac{1}{\mu_0} \nabla \mathbf{B}^2 + \tfrac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}}_{= \mathbf{J} \times \mathbf{B}} + \rho \mathbf{g} + \ldots$$

does $\mathbf{B} \cdot \nabla \mathbf{B}$ point?

A. Along ***B***
B. Perpendicular to ***B***
C. Perpendicular to ***J***
D. Neither of these
E. Perp to both ***J*** and ***B***

Total time derivative

Consider momentum equation

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g} + \ldots$$

is correct if $d\mathbf{u}/dt$ is evaluated on comoving fluid particle
→ Lagrangian derivative or material derivative

In a fixed frame, we have $\boldsymbol{u}(t,\boldsymbol{x})$

chain rule

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{dx_j}{dt}\frac{\partial \mathbf{u}}{\partial x_j} = \frac{\partial \mathbf{u}}{\partial t} + u_j \frac{\partial \mathbf{u}}{\partial x_j} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

"directional derivative" - similar to tension force $\mathbf{B} \cdot \nabla \mathbf{B}$

…also called advective derivative

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \equiv \frac{D\mathbf{u}}{Dt}$$

Seen before?

A. Yes
B. Maybe
C. Probably not
D. Certainly not

What about continuity equation?

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

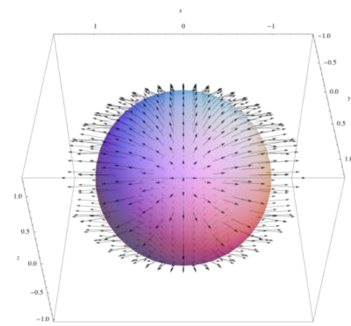
$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\oint_{\partial V} \rho \mathbf{u} \cdot d\mathbf{S}$$

change of mass in volume *V*

What gets in & out of surface ∂V

Seen before?

A. Yes
B. Maybe
C. Probably not
D. Certainly not



Recall Lecture 9

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$$B_y(x,t) = \hat{B}_y e^{ik_x x - i\omega t} + \text{c.c.}$$

$$E_z(x,t) = \hat{E}_z e^{ik_x x - i\omega t} + \text{c.c.}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} 0 \\ B_y(x,t) \\ 0 \end{pmatrix} = -\begin{pmatrix} \partial_x \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ E_z \end{pmatrix} = \partial_x E_z$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\begin{pmatrix} 0 \\ 0 \\ E_z(x,t) \end{pmatrix} = +\begin{pmatrix} \partial_x \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ B_y \\ 0 \end{pmatrix} = \partial_x B_y$$

Vanishing determinant

$$\det\begin{pmatrix} \omega & k_x \\ k_x / \mu_0 \varepsilon_0 & \omega \end{pmatrix} = 0$$

$$-i\omega \hat{B}_y = ik_x \hat{E}_z$$

$$-i\omega \hat{E}_z = ik_x \hat{B}_y / \mu_0 \varepsilon_0$$

Eigenvalue problem with eigenvalue ω

$$\omega = \pm k_x / \sqrt{\mu_0 \varepsilon_0}$$

Speed of light

$$c = 1/\sqrt{\mu_0 \varepsilon_0}$$

Combined with momentum equation

Expand continuity eqn: $$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) = -\mathbf{u} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{u}$$

Momntum eqn (isothermal): $$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\Re T}{\mu} \nabla \rho + \ldots$$

Linearized form

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \mathbf{u}_1$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\frac{\Re T}{\mu} \nabla \rho_1$$

Trial solution

$$\rho_1(z,t) = \hat{\rho}_1 e^{ik_z z - i\omega t} + \text{c.c.}$$

$$u_{1z}(z,t) = \hat{u}_{1z} e^{ik_z z - i\omega t} + \text{c.c.}$$

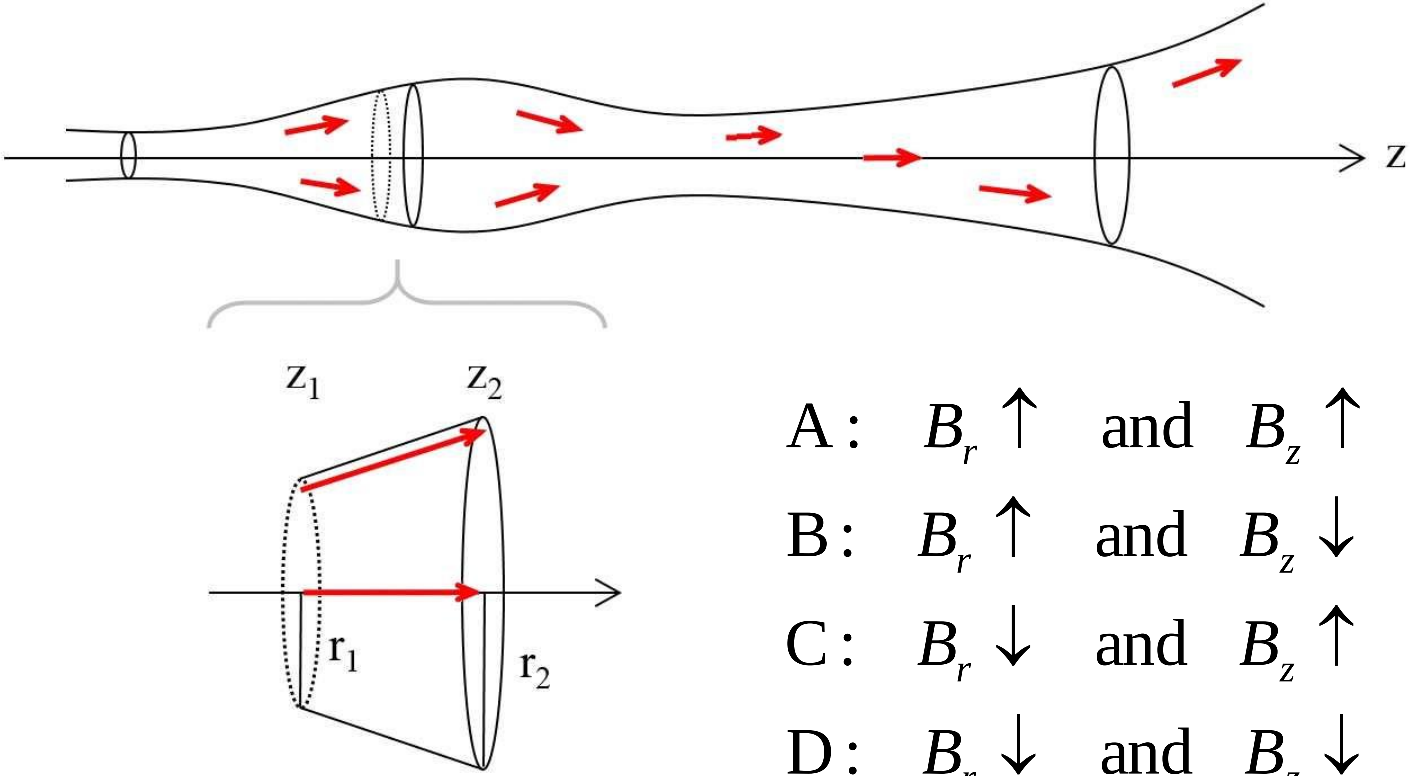
$$\begin{pmatrix} i\omega & -ik_z \rho_0 \\ -ik_z \dfrac{\Re T}{\mu} & i\omega \rho_0 \end{pmatrix} \begin{pmatrix} \hat{\rho}_1 \\ \hat{u}_{1z} \end{pmatrix} = 0$$

Dispersion relation $$\omega^2 = \frac{\Re T}{\mu} k_z^2$$

$$c_s = \sqrt{\Re T / \mu}$$ Sound speed

Static flux tube

$$\nabla \cdot \mathbf{B} = 0$$



A: $B_r \uparrow$ and $B_z \uparrow$

B: $B_r \uparrow$ and $B_z \downarrow$

C: $B_r \downarrow$ and $B_z \uparrow$

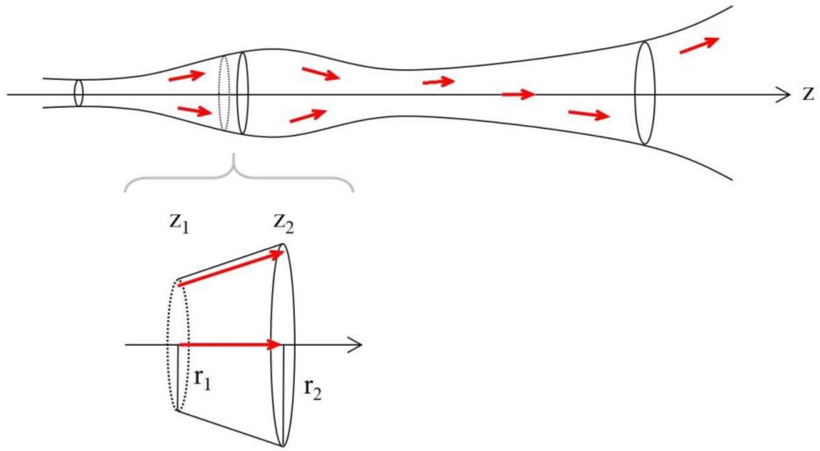
D: $B_r \downarrow$ and $B_z \downarrow$

Axisymmetry

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_r\right) + \frac{1}{r}\frac{\partial}{\partial \phi}B_\phi + \frac{\partial}{\partial z}B_z = 0$$

Boundary conditions on the axis (r=0)?



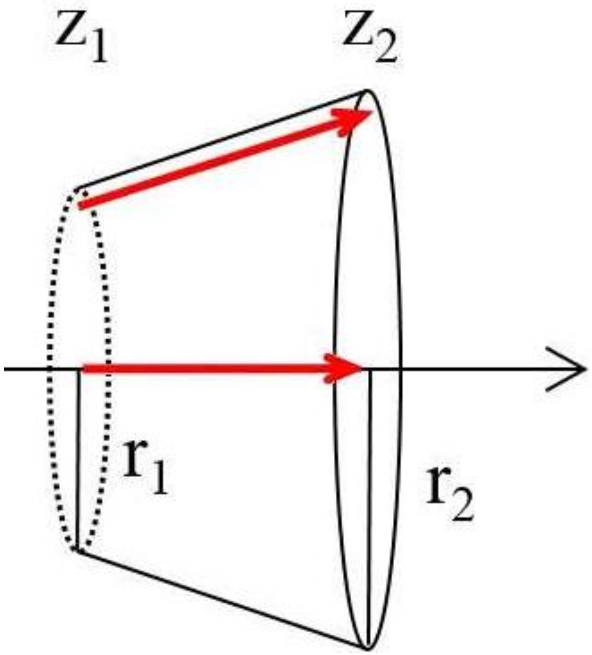
A: $B_r = 0$ and $B_z = 0$

B: $\frac{\partial}{\partial r} B_r = 0$ and $B_z = 0$

C: $B_r = 0$ and $\frac{\partial}{\partial r} B_z = 0$

D: $\frac{\partial}{\partial r} B_r = 0$ and $\frac{\partial}{\partial r} B_z = 0$

Flux conservation



$$\frac{B_{r1}}{B_{z1}} = \frac{\Delta r}{\Delta z}$$ (tube geometry)

$$B_{r1} = -\tfrac{1}{2}\frac{\Delta B_z}{\Delta z} r_1$$ (problem 3a)

Divide by each other $$\frac{\Delta B_z}{B_{z1}} = -2\frac{\Delta r}{r_1} = -\frac{\Delta A}{A}$$ (A=area)