# Magnetorotational Instability for inclined fields

## 1 Governing equations

In the presence of rotation and shear, the MHD equations for the departure from the shear flow $\boldsymbol{u}$, the magnetic field $\boldsymbol{B}$, and the density $\rho$ becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + Sx\frac{\partial \boldsymbol{u}}{\partial y} + u_x S\hat{\boldsymbol{y}} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\rho^{-1}\boldsymbol{\nabla}P + \rho^{-1}\boldsymbol{J}\times\boldsymbol{B}, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + Sx\frac{\partial \boldsymbol{B}}{\partial y} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = B_x S\hat{\boldsymbol{y}} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + Sx\frac{\partial \rho}{\partial y} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{3}$$

Let us here, for simplicity, consider an isothermal equation of state, i.e., $P=\rho c_\mathrm{s}^2$, where $c_\mathrm{s}=\mathrm{const}$. The equations can be readily linearized about $\boldsymbol{u}=0$, $\boldsymbol{B}=\boldsymbol{B}_0=\mathrm{const}$, and $\rho=\rho_0=\mathrm{const}$. For the following, we assume $\boldsymbol{B}_0=(0,B_{0y},B_{0z})$ and $\boldsymbol{\nabla}=(0,0,\partial_z)$. We assume that all perturbations are proportional to $e^{\sigma t+\mathrm{i}kz}$. Thus we have

$$\begin{pmatrix} \sigma & -2\Omega & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} & 0 \\ S+2\Omega & \sigma & 0 & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ 0 & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & 0 & \mathrm{i}k & \sigma & 0 & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 & \sigma & 0 \\ 0 & -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_{x1} \\ \hat{u}_{y1} \\ \hat{u}_{z1} \\ \hat{\rho}_1/\rho_0 \\ \hat{B}_{x1} \\ \hat{B}_{y1} \end{pmatrix} = 0. \tag{4}$$

We can split the determinant into 3 parts, $T_1+T_2+T_3=0$ with

$$T_1 = \sigma\det\begin{pmatrix} \sigma & 0 & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & \mathrm{i}k & \sigma & 0 & 0 \\ 0 & 0 & 0 & \sigma & 0 \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \tag{5}$$

$$T_2 = 2\Omega\det\begin{pmatrix} S+2\Omega & 0 & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & \mathrm{i}k & \sigma & 0 & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & \sigma & 0 \\ 0 & \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \tag{6}$$

$$T_3 = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\det\begin{pmatrix} S+2\Omega & \sigma & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ 0 & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & 0 & \mathrm{i}k & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & \sigma \end{pmatrix} \tag{7}$$

The first part becomes

$$T_1 = \sigma^2\det\begin{pmatrix} \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ \mathrm{i}k & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\sigma\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 \\ 0 & \mathrm{i}k & \sigma & 0 \\ 0 & 0 & 0 & \sigma \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix} \tag{8}$$

The second part becomes

$$T_2 = 2\Omega(S+2\Omega)\det\begin{pmatrix} \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ \mathrm{i}k & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}2\Omega\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 \\ 0 & \mathrm{i}k & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & \sigma \\ 0 & \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix} \quad (9)$$

and the third part is in turn split into 3 parts, $T_3 = T_{3\mathrm{a}} + T_{3\mathrm{b}} + T_{3\mathrm{c}}$ with

$$T_{3\mathrm{a}} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}(S+2\Omega)\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & \mathrm{i}k & \sigma & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & \sigma \end{pmatrix} \quad (10)$$

$$T_{3\mathrm{b}} = \sigma\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & \mathrm{i}k & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 \\ 0 & \mathrm{i}kB_{0y} & 0 & \sigma \end{pmatrix} \quad (11)$$

$$T_{3\mathrm{c}} = -\left(k\frac{B_{0z}}{\mu_0\rho_0}\right)^2\det\begin{pmatrix} 0 & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 \\ 0 & 0 & \mathrm{i}k & \sigma \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 \\ 0 & -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 \end{pmatrix} \quad (12)$$

## 1.1 $T_1$

Let us start with $T_1 = T_{1\mathrm{a}} + T_{1\mathrm{b}}$ so

$$T_{1\mathrm{a}} = \sigma^2\det\begin{pmatrix} \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ \mathrm{i}k & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \quad (13)$$

$$T_{1\mathrm{a}} = \sigma^3\det\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & -S & \sigma \end{pmatrix} - \mathrm{i}kc_\mathrm{s}^2\sigma^2\det\begin{pmatrix} \mathrm{i}k & 0 & 0 \\ 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & -S & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0}\sigma^2\det\begin{pmatrix} \mathrm{i}k & \sigma & 0 \\ 0 & 0 & \sigma \\ \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix} \quad (14)$$

so

$$T_{1\mathrm{a}} = \sigma^6 + k^2c_\mathrm{s}^2\sigma^4 + k^2\frac{B_{0y}^2}{\mu_0\rho_0}\sigma^4 = \sigma^6 + (\omega_\mathrm{c}^2 + \omega_{\mathrm{A}y}^2)\sigma^4 \quad (15)$$

and for $T_{1\mathrm{b}}$ we have

$$T_{1\mathrm{b}} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\sigma\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 \\ 0 & \mathrm{i}k & \sigma & 0 \\ 0 & 0 & 0 & \sigma \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix} \quad (16)$$

with

$$T_{1\mathrm{b}} = \mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\sigma^2\det\begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & \sigma \\ -\mathrm{i}kB_{0z} & 0 & -S \end{pmatrix} + k^2\frac{B_{0z}}{\mu_0\rho_0}\sigma c_\mathrm{s}^2\det\begin{pmatrix} 0 & \mathrm{i}k & 0 \\ 0 & 0 & \sigma \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & -S \end{pmatrix} \quad (17)$$

or

$$T_{1\mathrm{b}} = k^2\frac{B_{0z}^2}{\mu_0\rho_0}\sigma^4 + k^4\frac{B_{0z}^2}{\mu_0\rho_0}\sigma^2c_\mathrm{s}^2 = \omega_{\mathrm{A}z}^2\sigma^2(\sigma^2 + \omega_\mathrm{c}^2) \quad (18)$$

So in summary

$$T_1 = \sigma^6 + \omega_\mathrm{s}^2\sigma^4 + \omega_{\mathrm{A}y}^2\sigma^4 + \omega_{\mathrm{A}z}^2\sigma^4 + \omega_{\mathrm{A}z}^2\sigma^2\omega_\mathrm{s}^2 \quad (19)$$

## 1.2 $T_2$

Next, we work with $T_2 = T_{2\mathrm{a}} + T_{2\mathrm{b}}$ so

$$T_{2a} = 2\Omega(S+2\Omega)\det\begin{pmatrix} \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ \mathrm{i}k & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \tag{20}$$

so

$$T_{2a} = 2\Omega(S+2\Omega)\left[\sigma\det\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & -S & \sigma \end{pmatrix} - \mathrm{i}kc_\mathrm{s}^2\det\begin{pmatrix} \mathrm{i}k & 0 & 0 \\ 0 & \sigma & 0 \\ \mathrm{i}kB_{0y} & -S & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0}\det\begin{pmatrix} \mathrm{i}k & \sigma & 0 \\ 0 & 0 & \sigma \\ \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix}\right] \tag{21}$$

and thus

$$T_{2a} = 2\Omega(S+2\Omega)\left(\sigma^4 + k^2c_\mathrm{s}^2\sigma^2 + k^2\frac{B_{0y}^2}{\mu_0\rho_0}\sigma^2\right) \tag{22}$$

while

$$T_{2b} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}2\Omega\det\begin{pmatrix} 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 \\ 0 & \mathrm{i}k & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & \sigma \\ 0 & \mathrm{i}kB_{0y} & 0 & -S \end{pmatrix} \tag{23}$$

so

$$T_{2b} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}2\Omega\left[-\sigma\det\begin{pmatrix} 0 & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & \sigma \\ 0 & 0 & -S \end{pmatrix} + \mathrm{i}kc_\mathrm{s}^2\det\begin{pmatrix} 0 & \mathrm{i}k & 0 \\ -\mathrm{i}kB_{0z} & 0 & \sigma \\ 0 & \mathrm{i}kB_{0y} & -S \end{pmatrix}\right] \tag{24}$$

and thus

$$T_{2b} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}2\Omega\left(+\sigma^2S\mathrm{i}kB_{0z} + k^2c_\mathrm{s}^2S\,\mathrm{i}kB_{0z}\right) \tag{25}$$

or

$$T_{2b} = k^2\frac{B_{0z}^2}{\mu_0\rho_0}2\Omega S\left(\sigma^2 + k^2c_\mathrm{s}^2\right) \tag{26}$$

So in summary.

$$T_2 = 2\Omega(S+2\Omega)\left(\sigma^4 + \omega_\mathrm{s}^2\sigma^2 + \omega_{\mathrm{A}y}^2\sigma^2\right) + 2\Omega S\omega_{\mathrm{A}z}^2\left(\sigma^2 + \omega_\mathrm{s}^2\right) \tag{27}$$

## 1.3 $T_3$

And finally, we have

$$T_{3\mathrm{a}} = -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}(S+2\Omega)\left[-\sigma\det\begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & 0 \\ -\mathrm{i}kB_{0z} & 0 & \sigma \end{pmatrix}\right. \tag{28}$$

$$\left. + \mathrm{i}kc_\mathrm{s}^2\det\begin{pmatrix} 0 & \mathrm{i}k & 0 \\ 0 & 0 & 0 \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0}\det\begin{pmatrix} 0 & \mathrm{i}k & \sigma \\ 0 & 0 & 0 \\ -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 \end{pmatrix}\right] \tag{29}$$

but there are only zeros in the second row of all matrices, so

$$T_{3\mathrm{a}} = 0. \tag{30}$$

Next,

$$T_{3\mathrm{b}} = \sigma\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0}\left[-\sigma\det\begin{pmatrix} 0 & \sigma & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}\right. \tag{31}$$

$$\left. + \mathrm{i}kc_\mathrm{s}^2\det\begin{pmatrix} 0 & \mathrm{i}k & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 \\ 0 & \mathrm{i}kB_{0y} & \sigma \end{pmatrix} - \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0}\det\begin{pmatrix} 0 & \mathrm{i}k & \sigma \\ -\mathrm{i}kB_{0z} & 0 & 0 \\ 0 & \mathrm{i}kB_{0y} & 0 \end{pmatrix}\right], \tag{32}$$

so

$$T_{3\mathrm{b}} = \sigma \mathrm{i} k \frac{B_{0z}}{\mu_0\rho_0} \left[ -\sigma^3 \mathrm{i} k B_{0z} - \mathrm{i} k c_\mathrm{s}^2 \sigma k^2 B_{0z} - \mathrm{i} k \frac{B_{0y}}{\mu_0\rho_0} \sigma k^2 B_{0z} B_{0y} \right] \tag{33}$$

or

$$T_{3\mathrm{b}} = \sigma^2 k^2 \frac{B_{0z}^2}{\mu_0\rho_0} \left[ \sigma^2 + c_\mathrm{s}^2 k^2 + \frac{B_{0y}^2}{\mu_0\rho_0} k^2 \right] \tag{34}$$

And then

$$T_{3\mathrm{c}} = -\left( k \frac{B_{0z}}{\mu_0\rho_0} \right)^2 \left[ \sigma \det \begin{pmatrix} 0 & 0 & \sigma \\ -\mathrm{i} k B_{0z} & 0 & 0 \\ 0 & -\mathrm{i} k B_{0z} & 0 \end{pmatrix} - \mathrm{i} k c_\mathrm{s}^2 \det \begin{pmatrix} 0 & 0 & \mathrm{i} k \\ -\mathrm{i} k B_{0z} & 0 & 0 \\ 0 & -\mathrm{i} k B_{0z} & \mathrm{i} k B_{0y} \end{pmatrix} \right] \tag{35}$$

or

$$T_{3\mathrm{c}} = -\left( k \frac{B_{0z}}{\mu_0\rho_0} \right)^2 \left[ -\sigma^2 k^2 B_{0z}^2 - k^2 c_\mathrm{s}^2 k^2 B_{0z}^2 \right] \tag{36}$$

or

$$T_{3\mathrm{c}} = \left( k^2 \frac{B_{0z}^2}{\mu_0\rho_0} \right)^2 \left[ \sigma^2 + k^2 c_\mathrm{s}^2 \right] \tag{37}$$

So in summary.

$$T_3 = \sigma^2 \omega_{\mathrm{A}z}^2 (\sigma^2 + \omega_\mathrm{s}^2 + \omega_{\mathrm{A}y}^2) + \omega_{\mathrm{A}z}^4 (\sigma^2 + \omega_\mathrm{s}^2) \tag{38}$$

## 1.4 $T_1 + T_2 + T_3$

$$T_1 = \sigma^6 + \omega_\mathrm{s}^2 \sigma^4 + \omega_{\mathrm{A}y}^2 \sigma^4 + \omega_{\mathrm{A}z}^2 \sigma^4 + \omega_{\mathrm{A}z}^2 \sigma^2 \omega_\mathrm{s}^2 \tag{39}$$

$$T_2 = 2\Omega(S + 2\Omega) \left( \sigma^4 + \omega_\mathrm{s}^2 \sigma^2 + \omega_{\mathrm{A}y}^2 \sigma^2 \right) + 2\Omega S \omega_{\mathrm{A}z}^2 \left( \sigma^2 + \omega_\mathrm{s}^2 \right) \tag{40}$$

$$T_3 = \sigma^2 \omega_{\mathrm{A}z}^2 (\sigma^2 + \omega_\mathrm{s}^2 + \omega_{\mathrm{A}y}^2) + \omega_{\mathrm{A}z}^4 (\sigma^2 + \omega_\mathrm{s}^2) \tag{41}$$

So

$$T = \sigma^6 + \sigma^4 [\omega_\mathrm{s}^2 + \omega_{\mathrm{A}y}^2 + 2\omega_{\mathrm{A}z}^2 + 2\Omega(S + 2\Omega)] \tag{42}$$

$$+\sigma^2 \left[ 2\Omega(S + 2\Omega)(\omega_\mathrm{s}^2 + \omega_{\mathrm{A}y}^2) + \omega_{\mathrm{A}z}^2 (2\omega_\mathrm{s}^2 + \omega_{\mathrm{A}y}^2 + \omega_{\mathrm{A}z}^2 + 2\Omega S) \right] \tag{43}$$

$$+\omega_\mathrm{s}^2 \omega_{\mathrm{A}z}^2 (\omega_{\mathrm{A}z}^2 + 2\Omega S) \tag{44}$$

# References


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