ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 2: KEY (Due Fri., Sept 23, 2016) September 7, 2016, Axel Brandenburg

1. **MRI with inclined field.**

   (a) Show that the MRI with an inclined field, $\boldsymbol{B}_0 = (0, B_{0y}, B_{0z})$ and a spatial dependence of all variables only in the z direction is given by the following eigenvalue problem:

$$\begin{pmatrix} \sigma & -2\Omega & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} & 0 \\ S+2\Omega & \sigma & 0 & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ 0 & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & 0 & \mathrm{i}k & \sigma & 0 & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 & \sigma & 0 \\ 0 & -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_{x1} \\ \hat{u}_{y1} \\ \hat{u}_{z1} \\ \hat{\rho}_1/\rho_0 \\ \hat{B}_{x1} \\ \hat{B}_{y1} \end{pmatrix} = 0. \qquad (1)$$

   (b) Expand the matrix governing the eigenvalue problem and discuss the solutions. To simplify the notation, define the two Alfvén frequencies $\omega_{\mathrm{A}y}^2 = k^2 v_\mathrm{A}^2$ and $\omega_{\mathrm{A}z}^2 = k^2 v_\mathrm{A}^2$, with $v_\mathrm{A}^2 = B_{0z}^2/(\mu_0\rho_0)$ and v_A being the Alfvén velocity. as well as the acoustic frequency $\omega_\mathrm{c}^2 = k^2 c_\mathrm{s}^2$. Begin by discussing special cases: verify that the usual MRI solution is recovered for $B_{0y} = 0$.

   (c) Next, discuss $B_{0y} \neq 0$ and either (i) $B_{0z} = \Omega = S = 0$, (ii) $\Omega = S = 0$ with $B_{0z} \neq 0$, (iii) $S = 0$ with $\Omega \neq 0$ and $B_{0z} \neq 0$, and finally the case (iv) in which neither S, Ω, nor B_{0z} vanish,

   (d) Compute solutions of the full problem numerically. Plot first σ^2 as a function of the inclination θ of the field $\boldsymbol{B}_0 = B_0(0, \sin\theta, \cos\theta)$ for $\Omega = S = 0$ and $c_\mathrm{s} = 1$ and (i) $v_\mathrm{A} = 0.7$, (ii) $v_\mathrm{A} = 1$, (iii) $v_\mathrm{A} = 2$.

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   (a) In the presence of rotation and shear, the MHD equations for the departure from the shear flow $\boldsymbol{u}$, the magnetic field $\boldsymbol{B}$, and the density ρ becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + Sx\frac{\partial \boldsymbol{u}}{\partial y} + u_x S\hat{\boldsymbol{y}} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\rho^{-1}\boldsymbol{\nabla}P + \rho^{-1}\boldsymbol{J}\times\boldsymbol{B}, \qquad (2)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + Sx\frac{\partial \boldsymbol{B}}{\partial y} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = B_x S\hat{\boldsymbol{y}} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \qquad (3)$$

$$\frac{\partial \rho}{\partial t} + Sx\frac{\partial \rho}{\partial y} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}. \qquad (4)$$

   Let us here, for simplicity, consider an isothermal equation of state, i.e., $P = \rho c_\mathrm{s}^2$, where $c_\mathrm{s} = \mathrm{const}$. The equations can be readily linearized about $\boldsymbol{u} = 0$, $\boldsymbol{B} = \boldsymbol{B}_0 = \mathrm{const}$, and $\rho = \rho_0 = \mathrm{const}$. For the following, we assume $\boldsymbol{B}_0 = (0, B_{0y}, B_{0z})$ and $\boldsymbol{\nabla} = (0, 0, \partial_z)$. We assume that all perturbations are proportional to $e^{\sigma t + \mathrm{i}kz}$. Thus we have

$$\begin{pmatrix} \sigma & -2\Omega & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} & 0 \\ S+2\Omega & \sigma & 0 & 0 & 0 & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ 0 & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & \mathrm{i}k\frac{B_{0y}}{\mu_0\rho_0} \\ 0 & 0 & \mathrm{i}k & \sigma & 0 & 0 \\ -\mathrm{i}kB_{0z} & 0 & 0 & 0 & \sigma & 0 \\ 0 & -\mathrm{i}kB_{0z} & \mathrm{i}kB_{0y} & 0 & -S & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_{x1} \\ \hat{u}_{y1} \\ \hat{u}_{z1} \\ \hat{\rho}_1/\rho_0 \\ \hat{B}_{x1} \\ \hat{B}_{y1} \end{pmatrix} = 0. \qquad (5)$$

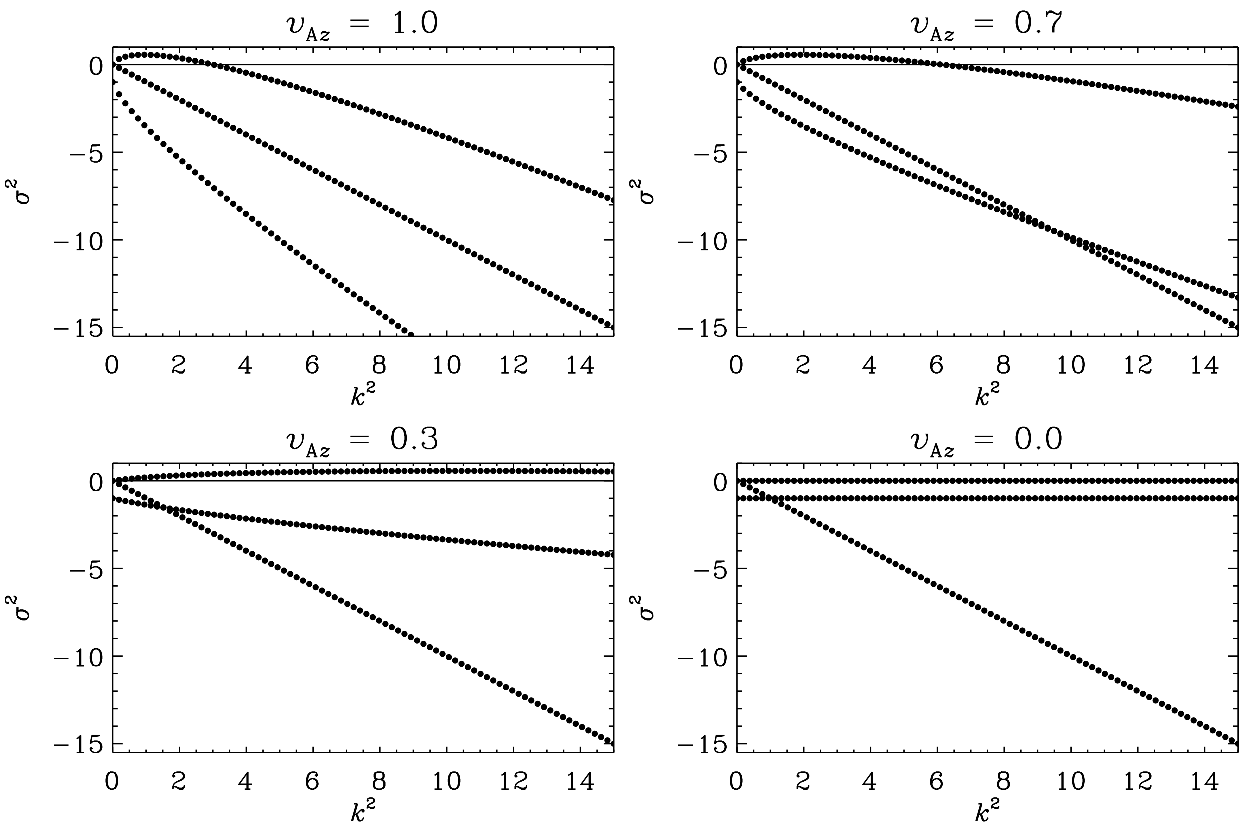

Figure 1: Starting from the usual MRI case (here $\Omega = 1$, $S = -3/2$, $v_{\mathrm{A}y} = 0$ and $c_{\mathrm{s}} = 1$), we see that for decreasing $v_{\mathrm{A}z}$ the lower (fast magnetosonic) branch becomes essentially flat and turns into what is called an inertial mode with $-\sigma^2 = \Omega$. The slow magnetosonic branch also becomes essentially flat, but with zero frequency.

Requiring the determinant to vanish yields the dispersion relation as

$$\sigma^6 + \sigma^4[\omega_{\mathrm{c}}^2 + \omega_{\mathrm{A}y}^2 + 2\omega_{\mathrm{A}z}^2 + 2\Omega(S + 2\Omega)] \tag{6}$$

$$+\sigma^2 \left[2\Omega(S + 2\Omega)(\omega_{\mathrm{c}}^2 + \omega_{\mathrm{A}y}^2) + \omega_{\mathrm{A}z}^2(2\omega_{\mathrm{c}}^2 + \omega_{\mathrm{A}y}^2 + \omega_{\mathrm{A}z}^2 + 2\Omega S)\right] \tag{7}$$

$$+\omega_{\mathrm{c}}^2\omega_{\mathrm{A}z}^2(\omega_{\mathrm{A}z}^2 + 2\Omega S) = 0 \tag{8}$$

(b) We already know that for $v_{\mathrm{A}y} = 0$, the soundwaves decouple. In that case the full MRI, as discussed in Handout 2, is recovered. In Fig. 1 we see the dispersion relation for different values of $v_{\mathrm{A}0}$. This case is interesting, because the Alfvén and fast magnetosonic modes cross at a certain value of k.

(c) For $v_{\mathrm{A}y} \neq 0$, however, soundwaves no longer decouple. An important limit is $\Omega = S = 0$, in which case we can write

$$\sigma^6 + \sigma^4(\omega_{\mathrm{c}}^2 + \omega_{\mathrm{A}y}^2 + 2\omega_{\mathrm{A}z}^2) + \sigma^2\omega_{\mathrm{A}z}^2(2\omega_{\mathrm{c}}^2 + \omega_{\mathrm{A}y}^2 + \omega_{\mathrm{A}z}^2) + \omega_{\mathrm{c}}^2\omega_{\mathrm{A}z}^4 = 0. \tag{9}$$

Inspecting again Eq. (5), we see that the first and fifth row and colum collapse into an independent matrix whose determinant must vanish, i.e.

$$\det \begin{pmatrix} \sigma & -\mathrm{i}k\frac{B_{0z}}{\mu_0\rho_0} \\ -\mathrm{i}kB_{0z} & \sigma \end{pmatrix} = 0, \tag{10}$$

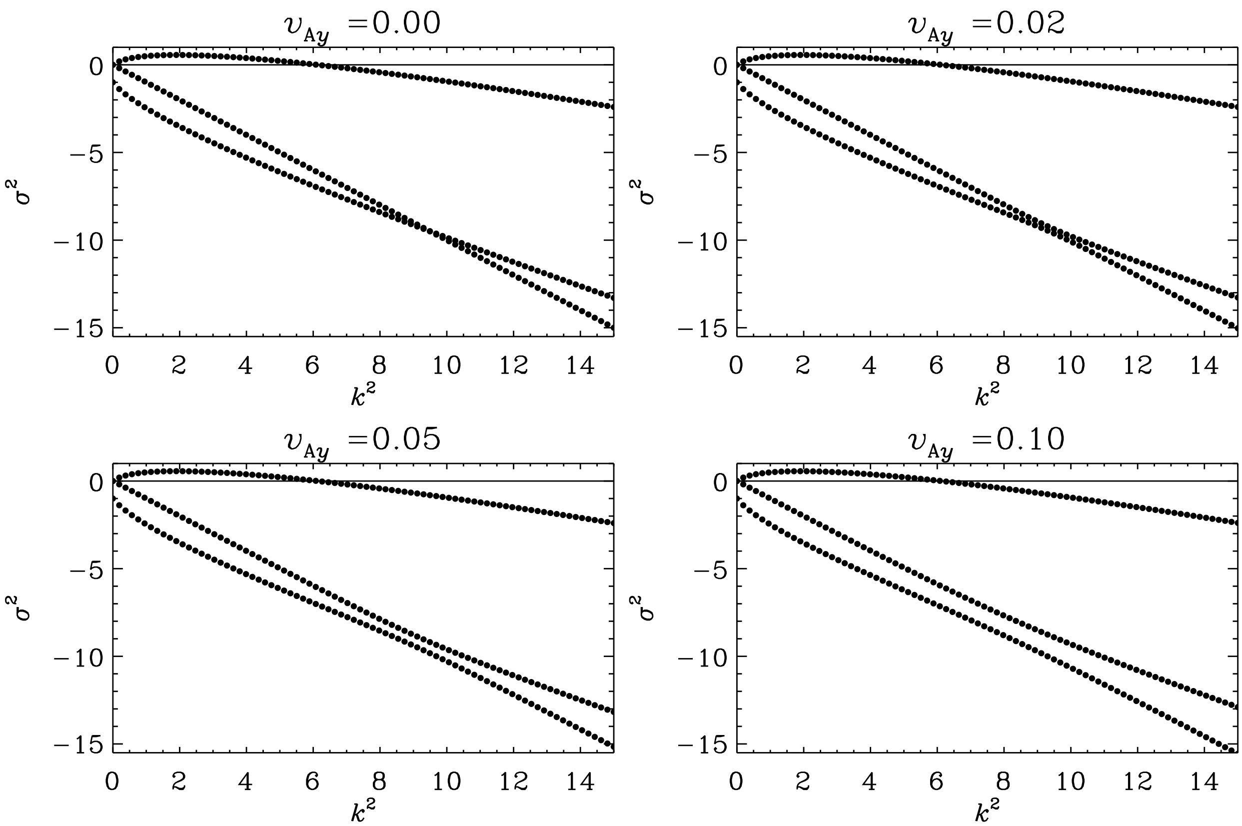

Figure 2: MRI case (again with $\Omega = 1$, $S = -3/2$, and $c_\mathrm{s} = 1$), but now with $v_{\mathrm{A}z} = 0.7$ and 4 values of $v_{\mathrm{A}y}$. Note the appearence of *avoided crossings*.

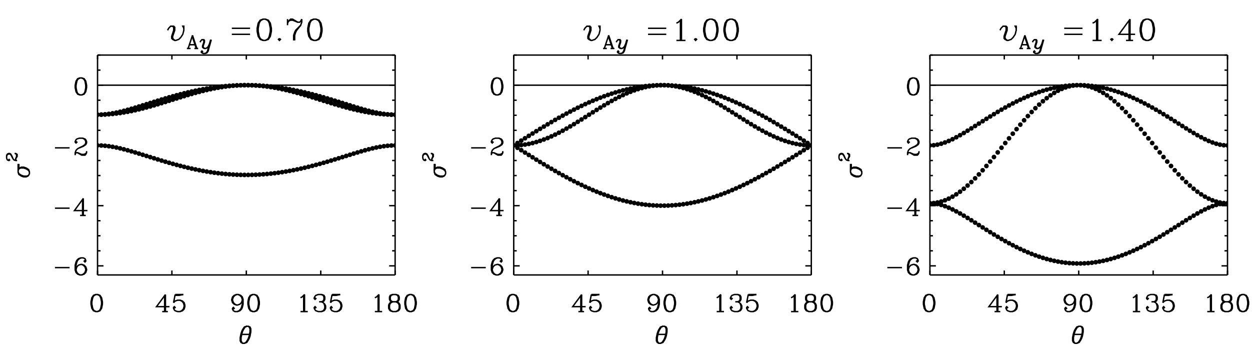

Figure 3: Dependence on θ for $\Omega = S = 0$, $c_\mathrm{s} = 1$, and three values of v_A.

so the dispersion relation reads

$$(\sigma^2 + \omega_{\mathrm{A}z}^2)\left[\sigma^4 + \sigma^2(\omega_\mathrm{c}^2 + \omega_{\mathrm{A}y}^2 + \omega_{\mathrm{A}z}^2) + \omega_{\mathrm{A}z}^2\omega_\mathrm{c}^2\right] = 0 \tag{11}$$

Note that the sound and fast magnetosonic branches *cross*; see Fig. 1. For finite values of $v_{\mathrm{A}y}$, these branches no longer cross. In such a case, one often talks about *avoided crossings*. The onset of MRI is not affected, however; see Fig. 2.

(d) We now consider the dependence on θ. Fig. 3 shows the three modes for $\Omega = S = 0$.

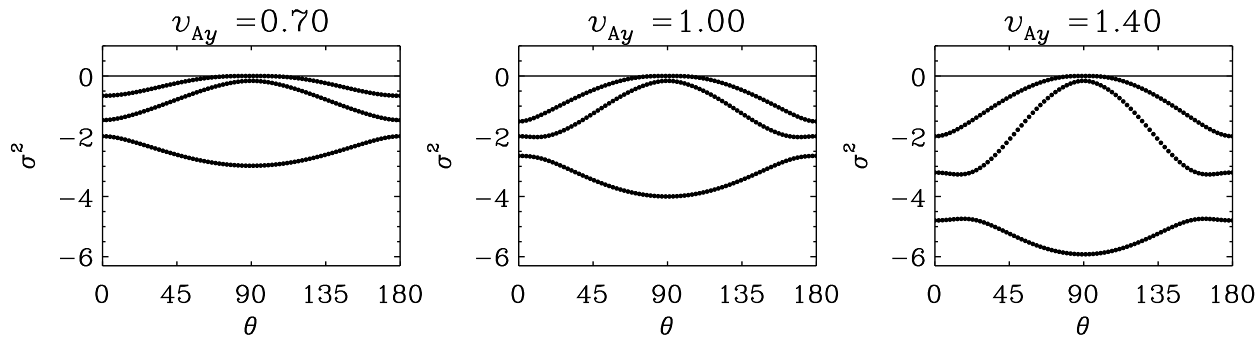

Figure 4: Similar to Fig. 3, but for $\Omega = 0.2$. Note that now the degeneracy between Alfvén and magnetosonic modes is lifted even for the fast magnetosonic modes at $\theta = 0°$ and the slow magnetosonic modes at $\theta = 90°$.

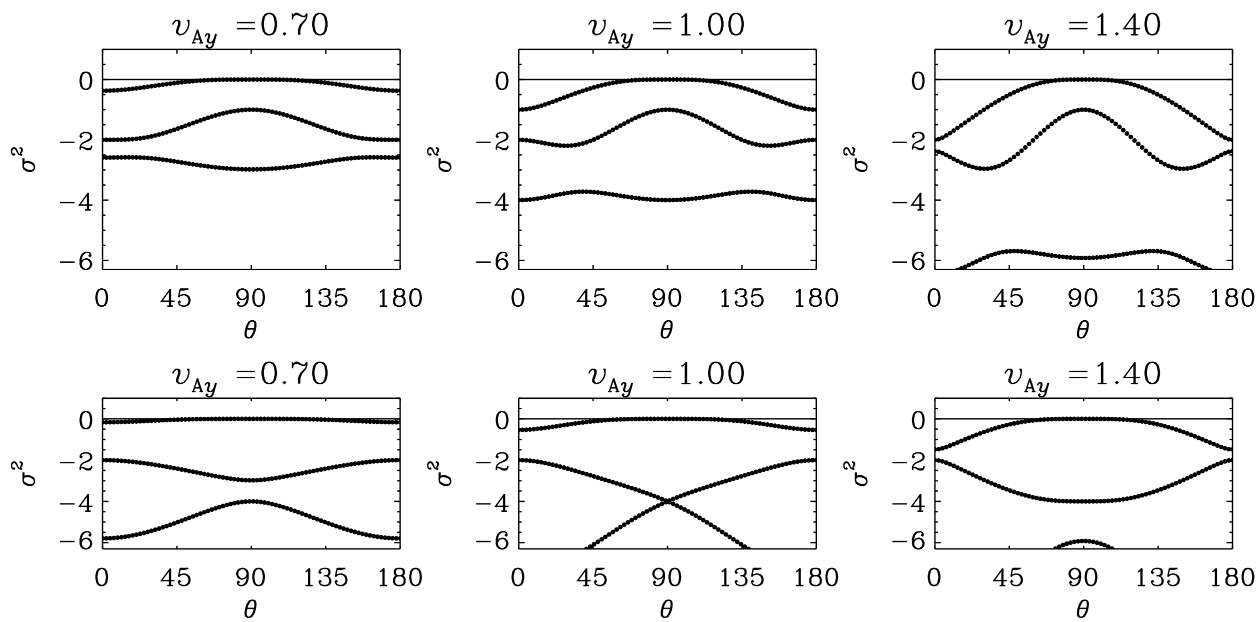

Figure 5: Similar to Fig. 3, but for $\Omega = 0.8$ (upper row) and $\Omega = 1$ (lower row). Note the dramatic changes between these two cases.

Note that the degeneracy between Alfvén and magnetosonic modes is lifted, except for the fast magnetosonic modes at $\theta = 0°$ and the slow magnetosonic modes at $\theta = 90°$. For finite values of vAy, the degeneracy between Alfvén and magnetosonic modes is lifted even for the fast magnetosonic modes at $\theta = 0°$ and the slow magnetosonic modes at $\theta = 90°$; see Fig. 4. As we increase the value of Ω, the θ dependence changes dramatically between all these case; see Fig. 5. Finally, for finite shear, we see the emergence of MRI for certain angles; see Fig. 6.

2. Compute numerically the solutions of the anharmonic oscillator

$$\ddot{x} = -\sin x \tag{12}$$

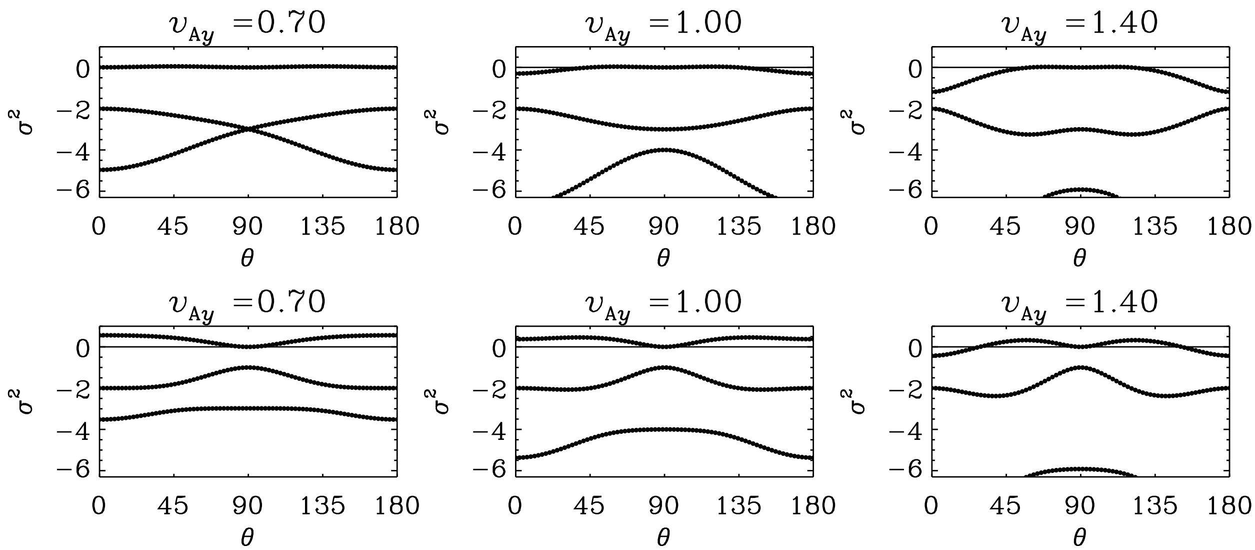

Figure 6: Similar to Fig. 5, but for $\Omega = 1$ and $S = -0.5$ (upper row) and $S = -1.5$ (lower row).

both as $x(t)$ and $\dot{x}(t)$, but also, for a set of different initial conditions, as parametric plots, in the plane $x(t)$ vs $\dot{x}(t)$.

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Let me show you here the very nice illustration by Michelle Maiden in Figs. 7 and 8.

3. Compute the eigenfrequencies of the Rayleigh-Benard problem with free-slip boundary conditions and negative values of Ra for parameters of your choice. Explain in words the physical difference between positive and negative values of Ra.

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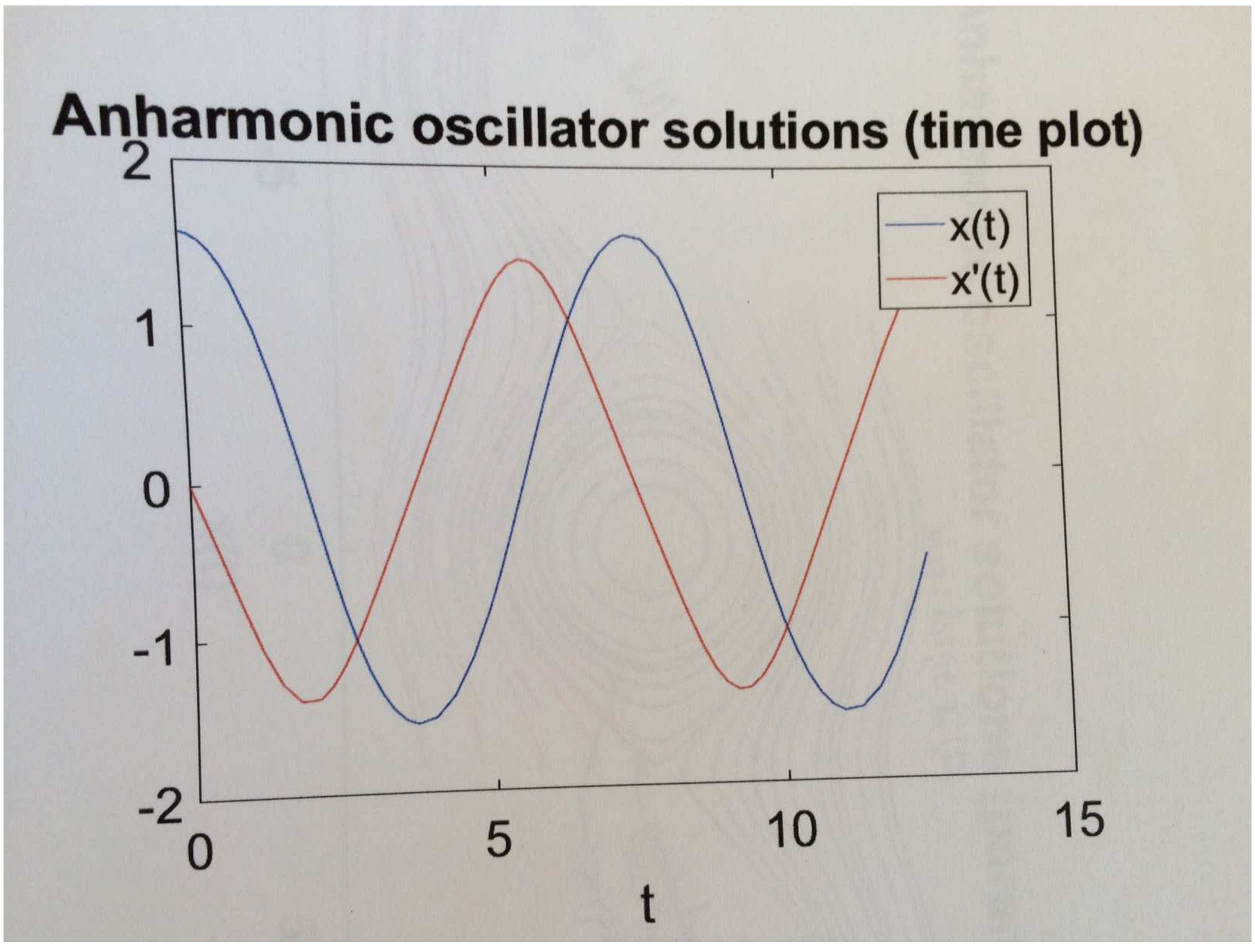
See Fig. 9. For negative Ra, there are only oscillatory solutions that corresond to Brunt-Väsälä oscillations; see Fig. 9.



Anharmonic oscillator solutions (time plot)
2
1
0
-1
-2
0
5
10
15
t
x(t)
x'(t)

Figure 7:

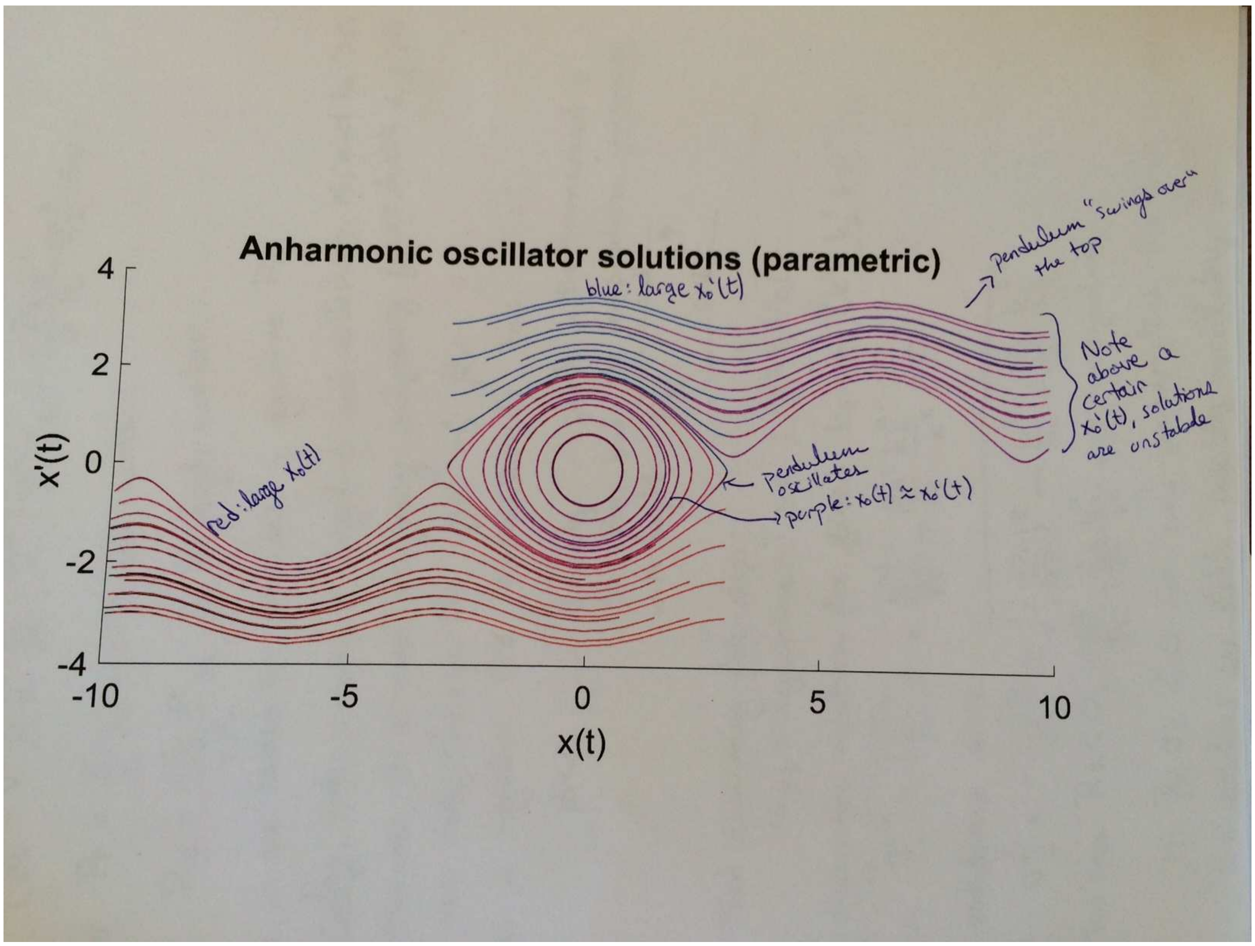

Anharmonic oscillator solutions (parametric)
x'(t)
x(t)
4
2
0
-2
-4
-10
-5
0
5
10
blue: large $x_0'(t)$
red: large $x_0(t)$
pendulum oscillates
purple: $x_0(t) \approx x_0'(t)$
pendulum "swings over the top"
Note above a certain $x_0'(t)$, solutions are unstable

Figure 8:

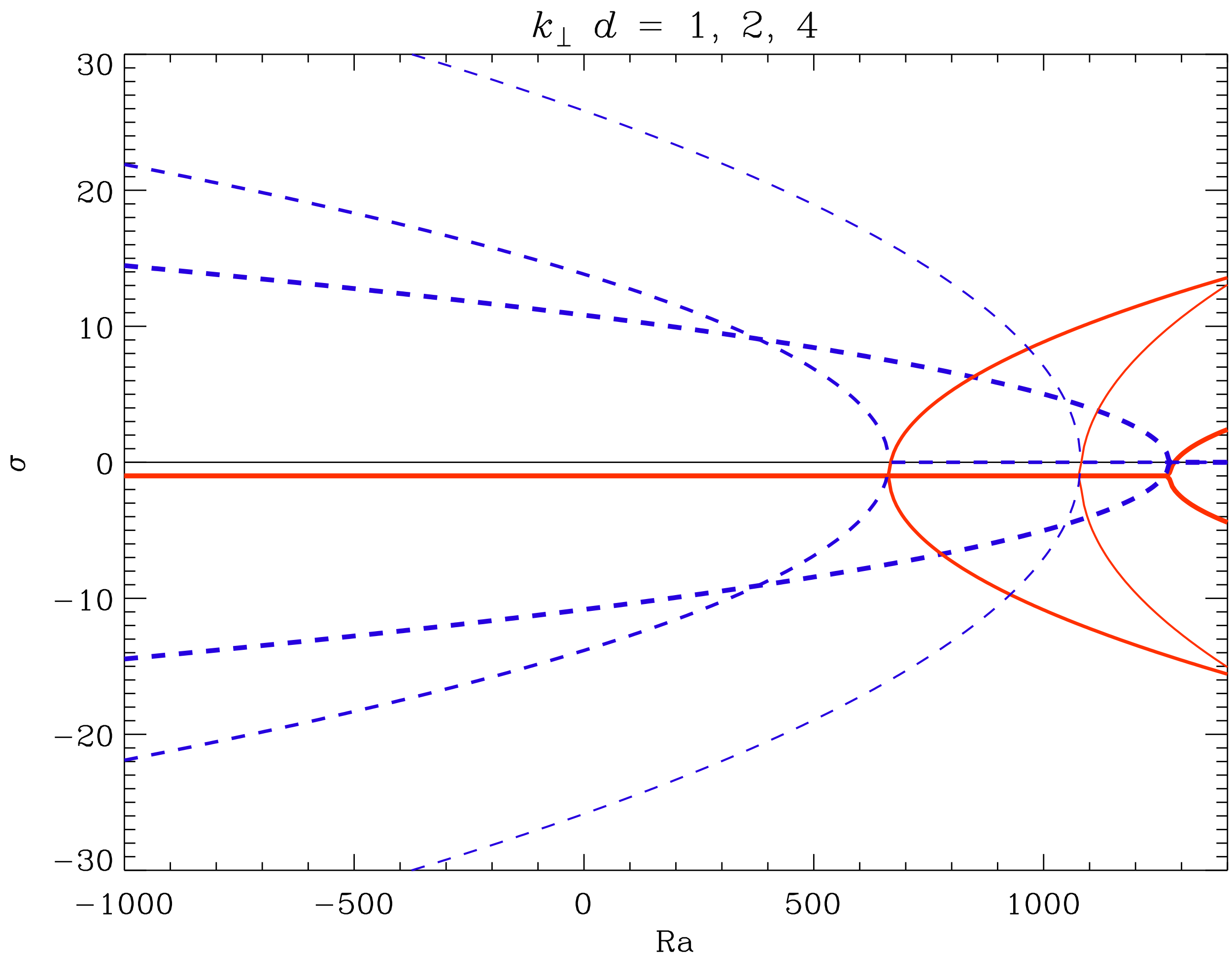


Figure 9: Similar to the plot in Handout 3.