ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 2: extra KEY (Sept 30, 2016)

2. Compute numerically the solutions of the anharmonic oscillator

$$\ddot{x} = -\sin x \tag{1}$$

both as $x(t)$ and $\dot{x}(t)$, but also, for a set of different initial conditions, as parametric plots, in the plane $x(t)$ vs $\dot{x}(t)$.

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The equations were solved numerically in double precision with a third-order Runge-Kutta scheme and fixed time step ($\delta t = 0.01$). The initial condition is chosen to be

$$x(0) = 0, \quad \dot{x}(0) = 2 + \epsilon. \tag{2}$$

In Fig. 1 the evolution is shown for three value of ϵ. For $\epsilon = -10^{-8}$ (red), the solution takes sharp turns and stays on a limit cycle (upper panel). The time evolution is markedly anharmonic. For $\epsilon = +10^{-7}$ (black), the solution escapes and jumps higher in x with each semi-orbit. This is referred to as a heteroclinic orbit[1]. However, and this is probably due to finite numerical accuracy, it does get stuck pretty soon and is then caught between $x = 3\pi$ and 5π; see the black line. For $\epsilon = +10^{-6}$ (thick-blue, dashed), the solution continues to escape. Note also the shorter period in the latter case.

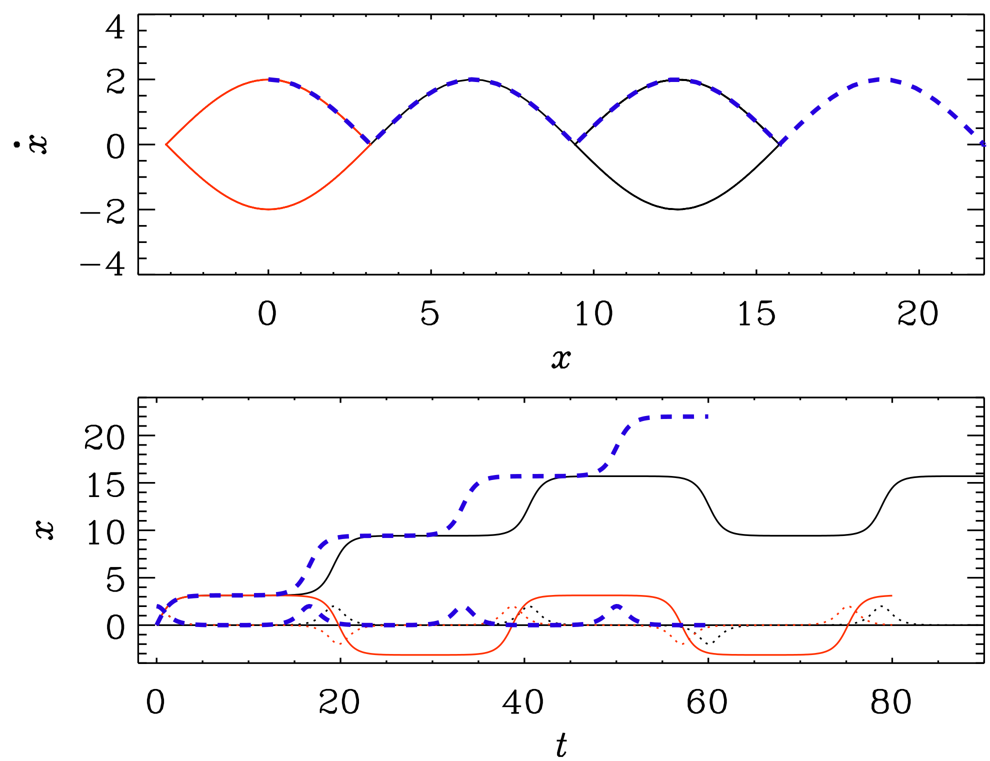

Figure 1: Parametric representation of $\dot{x}(t)$ versus $x(t)$ (upper panel) and time dependence of both $x(t)$ (solid or dashed) and $\dot{x}(t)$ (dotted).

[1] https://en.wikipedia.org/wiki/Heteroclinic_orbit

3. Compute the eigenfrequencies of the Rayleigh-Benard problem with free-slip boundary conditions and negative values of Ra for parameters of your choice. Explain in words the physical difference between positive and negative values of Ra.

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As pointed out by Loren, a k^2 term was missing in the original Handout 3. The corrected version was on my website since September 12. The corrected dispersion relation reads

$$\sigma_\pm = -\frac{1+\mathrm{Pr}}{2\,\mathrm{Pr}}k^2 \pm \sqrt{\frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}^2}k^4 - \frac{k^4}{\mathrm{Pr}} + \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{k_\perp^2}{k^2}}. \tag{3}$$

or

$$\sigma_\pm = -\frac{1+\mathrm{Pr}}{2\,\mathrm{Pr}}k^2 \pm \sqrt{\left(\frac{1-\mathrm{Pr}}{2\,\mathrm{Pr}}k^2\right)^2 + \frac{\mathrm{Ra}}{\mathrm{Pr}}\frac{k_\perp^2}{k^2}}. \tag{4}$$

See Fig. 2. For negative Ra, there are only oscillatory solutions that correspond to Brunt-Väsälä oscillations; see Fig. 2.

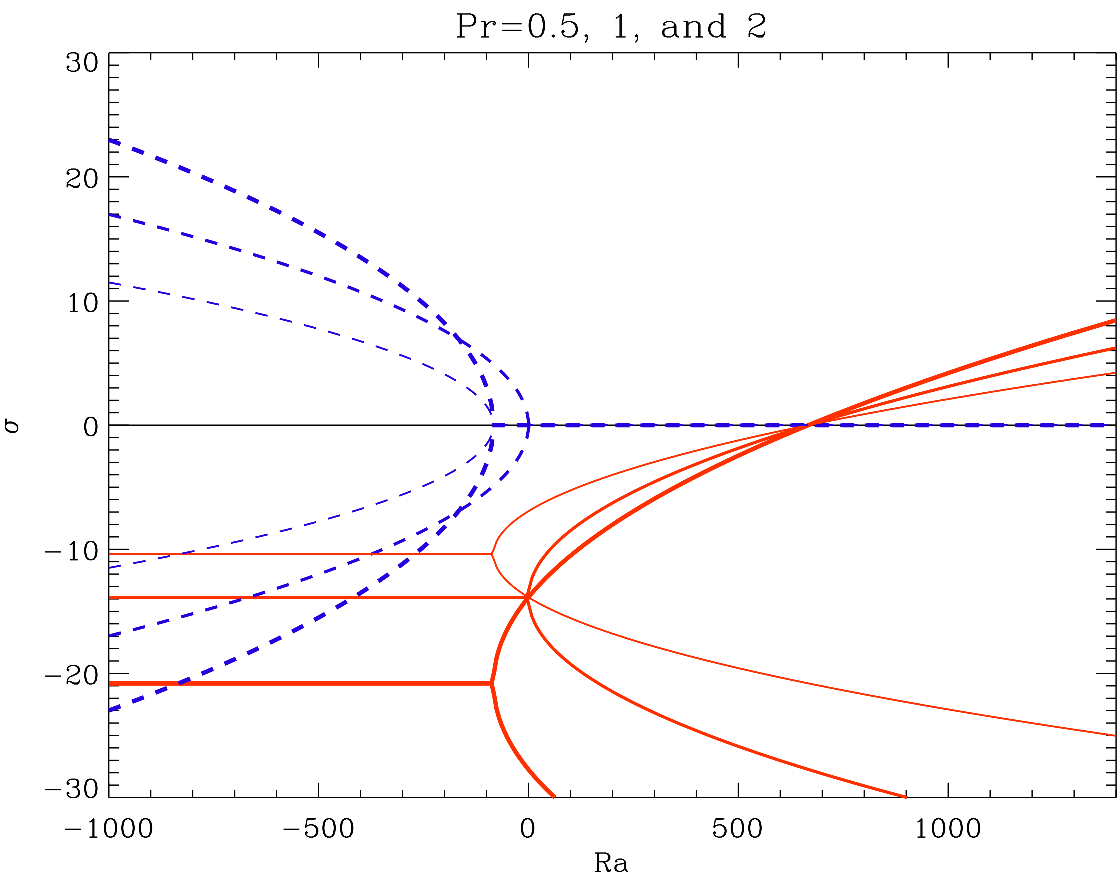

Figure 2: Similar to the plot in Handout 3, but now corrected. Fatter lines correspond to larger values of Pr_M.

It is interesting to note that only for $\mathrm{Pr}_M = 1$, oscillatory solutions occur for $\mathrm{Ra} < 0$. Both for larger and smaller values of Pr, there is an interval of negative values of Ra where no oscillations are possible. Mathematically, this is because $(\mathrm{Pr}-1)^2$ has a minimum at 1.