ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 3 (Due Wed., Oct 5, 2016) October 19, 2016, Axel Brandenburg

1. **Stability of different solution branches**

   Consider the following amplitude equation:

   $$\dot{\xi} = (R - R_{\rm c})\xi + \epsilon\xi^2 - \xi^3 \tag{1}$$

   Here, ξ is the amplitude, R is the control parameter. and $R_{\rm c}$ is the critical value for onset. Consider the parameters $R_{\rm c} = 1$ and $\epsilon = 1$.

   (a) Determine all fixed points of Eq. (1). Plot ξ vs. R. At this point, use only a dotted line for each of the solution branches. What is the nature of the bifurcation at $R = R_{\rm c}$? (Is it super- or subcritical?)

   (b) For each of these fixed points, linearize Eq. (1) around these fixed points and determine thereby the stability of these solutions to the full nonlinear equation on all branches. Now modify your plot and mark all stable solutions as a fat line.

   (c) Compute numerically the time dependence $\xi(t)$ for $R = 0.8$ using as initial conditions $\xi(0) = 0.2764$, 0.2763, 0.72, and 0.74. For each of the four cases, plot (i) $\xi = \xi(t)$, (ii) $|\xi(t) - \xi_0|$, where ξ_0 is the relevant fixed point solution, and finally (iii) the *instantaneous* growth rate, $\sigma(t) = {\rm d}\ln|\xi(t) - \xi_0|/{\rm d}t$, and determine the time interval in which $\sigma(t)$ can be used to estimate the growth or decay rate of the perturbed solution.

   (d) Compare the $\sigma(t)$ with the σ obtained under (b). Explain in a few words the reasons for discrepancies. Also, explain in a few words the structure of the bifurcation diagrams in terms of the slope of the branches. What do you think one should do with negative values of ξ? What happens if ϵ were negative?

   (e) Imagine applying the amplitude equation to a nonlinear laboratory dynamo problem, where dynamo action occurs when the magnetic Reynolds number has to be above a certain critical value for the dynamo to be excited. In the lab you would measure $\approx 0.5\,{\rm G}$ even in the absence of a dynamo because of the Earth's magnetic field. How do you think Eq. (1) needs to be modified to take this into account.

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   (a) Fixed points are solutions where $\dot{\xi} = 0$. Thus, we have

   $$0 = (R - R_{\rm c})\xi + \epsilon\xi^2 - \xi^3. \tag{2}$$

   There are 3 solutions; one of them is $\xi = \xi_0 = 0$ and the other two are given by

   $$\xi_\pm = \tfrac{1}{2}\epsilon \pm \sqrt{\tfrac{1}{4}\epsilon^2 + R - R_{\rm c}} \tag{3}$$

   A plot of ξ vs R is shown in the left panel of Fig. 1. For $\epsilon > 0$, the bifurcation is subcritical, because solutions are possible for $-\epsilon^2/4 < R - R_{\rm c} < 0$. In particular, for $R - R_{\rm c} = -0.2$, we have $\xi_+ \approx 0.723607$ (red dot) and $\xi_- \approx 0.276393$ (blue dot) we have solutions.

   (b) To perform a stability analysis, we linearize around any of the three fixed points. For the solution $\xi = \xi_0 = 0$, we know already that it becomes unstable for $R > R_{\rm c}$. To compute the stability for the other two solutions, we write

   $$\xi(t) = \xi_\pm + \xi_1(t) \tag{4}$$

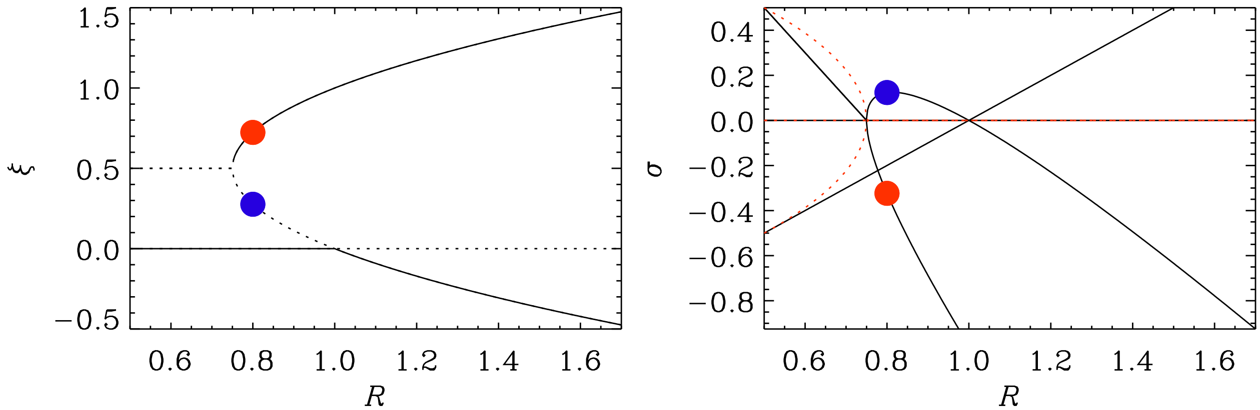

Figure 1: Bifurcation diagram for $R_\mathrm{c} = 1$ and $\epsilon = 1$.

and insert this into Eq. (1):

$$\dot{\xi}_1 = (R - R_\mathrm{c})(\xi_\pm + \xi_1) + \epsilon(\xi_\pm^2 + 2\xi_\pm\xi_1) - \xi_\pm^3 - 3\xi_\pm^2\xi_1. \tag{5}$$

We subtract the steady solutions, which we know obey $0 = (R - R_\mathrm{c})\xi_\pm + \epsilon\xi_\pm^2 - \xi_\pm^3$, so we are left with

$$\dot{\xi}_1 = (R - R_\mathrm{c})\xi_1 + 2\epsilon\xi_\pm\xi_1 - 3\xi_\pm^2\xi_1 \tag{6}$$

We assume the perturbations to be of the form $\xi_1(t) = \hat{\xi}\, e^{\sigma t}$, so we obtain the characteristic equation in the form

$$\sigma = (R - R_\mathrm{c}) + 2\epsilon\xi_\pm - 3\xi_\pm^2. \tag{7}$$

The solution, $\sigma = \sigma(R)$, is plotted in the right-hand panel of Fig. 1. All unstable branches are now marked as a dotted line in the left panel of Fig. 1. For $R = 0.8$, we have $\sigma(\xi_+) \approx -0.324$ and $\sigma(\xi_-) \approx +0.124$.

(c) We now solve Eq. (1) numerically for the initial conditions specified above. In Fig. 2 we show the evolution of $\xi(t)$, the departure from the equilibrium solution $|\xi(t) - \xi(0)|$, and the instantaneous growth rate, $\sigma(t) = \mathrm{d}\ln|\xi(t) - \xi_0|/\mathrm{d}t$, for the initial condition $\xi = 0.2764$. The reference slope of $\sigma = 0.12$ is also shown. Clearly, this solution is unstable, but the exponential departure from the equilibrium solution can clearly be seen for a *limited amount of time*. If one waits too long, a new (stable) equilibrium solution is obtained (upper branch, $\xi = \xi_+$), and so $\sigma(t)$ becomes zero, but this is now not a measure of the growth rate any more. Likewise, at early times, the growth is not yet exponential. This is because our initial condition was not close enough to the actual *eigenfunction* of the perturbed equation. The correct eigenfunction emerges a bit later automatically, because it is the fastest growing one.

In Fig. 3 we show the corresponding evolution for the initial condition $\xi = 0.2763$. Now we are initially slightly below the equilibrium solution. Again, the solution departs exponentially and reaches eventually the "trivial" solution, $\xi = \xi_0 = 0$.

Next, we consider the initial condition $\xi = 0.72$; see Fig. 4. This is close to the stable solution, so one has to go a certain distance away from the equilibrium solution to be able to see the exponential departure *toward* the final solution. Eventually one is so close to the final state that the small difference can no longer be resolved with the finite numerical accuracy, even in double precision.

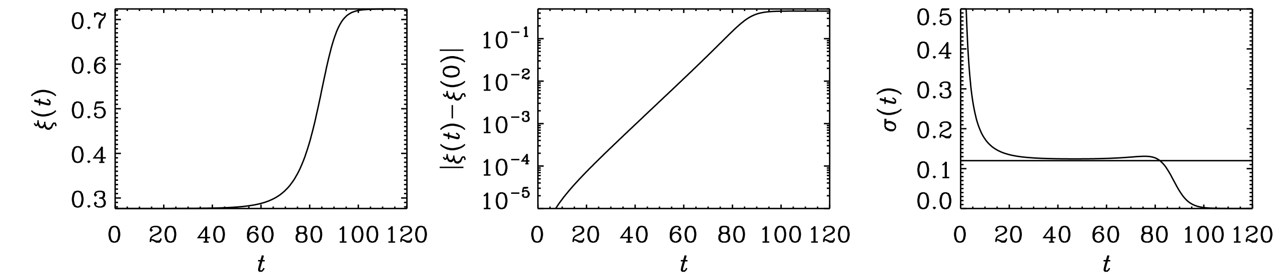

Figure 2: Initial condition $\xi = 0.2764$. Reference slope $\sigma = 0.12$.

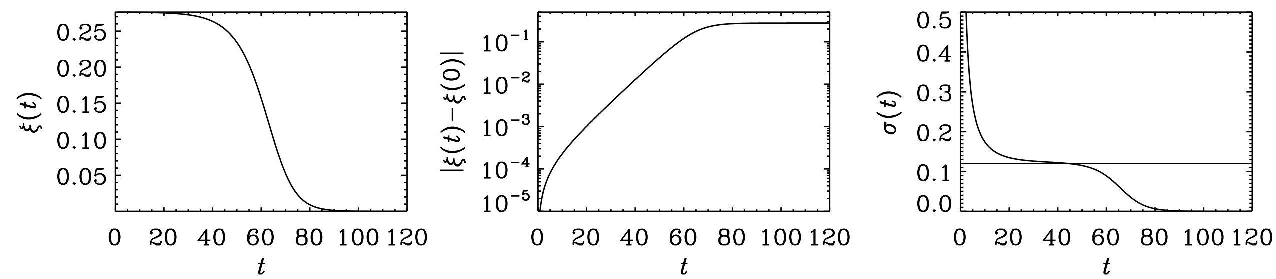

Figure 3: Initial condition $\xi = 0.2763$. Reference slope $\sigma = 0.12$.

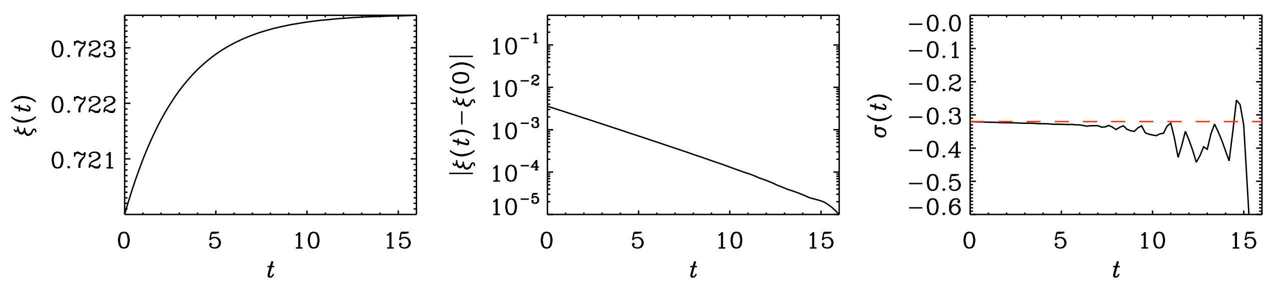

Figure 4: Initial condition $\xi = 0.72$. Reference slope $\sigma = -0.32$.

Finally, for the initial condition $\xi = 0.74$, we find similar behavior, but now ξ approaches the final solution from above; see Fig. 5.

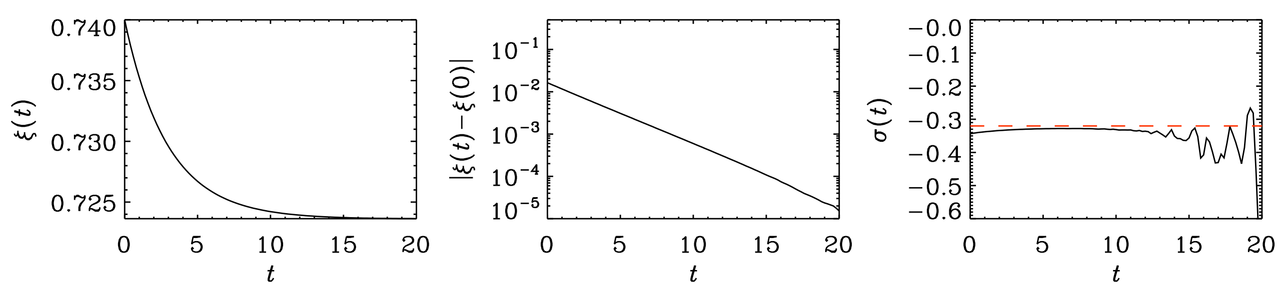

Figure 5: Initial condition $\xi = 0.74$. Reference slope $\sigma = -0.32$.

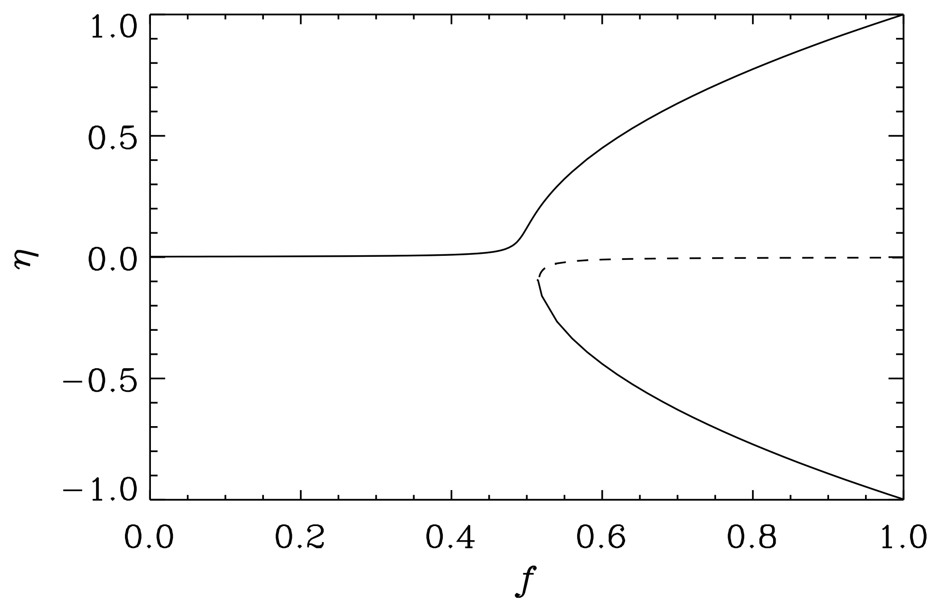

Figure 6: Enantiomeric excess η as a function of the strength of auto-catalysis f.

(d) As already explained above, the reason for departures between $\sigma(t)$ and the actual eigenvalue σ is that our numerical solutions reach the correct eigenfunction only after some time, and also only for a limited amount of time.

Unstable solutions (dotted lines in Fig. 1) have negative slopes in the bifurcation diagram, at least for $\xi > 0$.

In principle, negative values of ξ are perfectly fine, but if we are thinking of amplitude equations, we should discard those solutions.

When $\epsilon < 0$, the bifurcation at $R = R_c$ becomes supercritical.

(e) One obtains a so-called *imperfect bifurcation*. Fig. 6 shows such a plot in the context of astrobiology, where η is the so-called enantiomeric excess, i.e., the dominance of one handedness over the other[1] In this case, η is not just an amplitude, but a *signed quantity*: negative means left handedness, and positive right handedness. In astrobiology, a small preference of one handedness could come about by having (terrestrial) enzymes (=catalysts) giving rise to one particular handedness. Also the electroweak force breaks chirality, but that effect is very likely too small.

In the context of the Earth's magnetic field, if we looked at one component of the magnetic field (B_z, say), that would then also be a signed quantity, so positive and negative values could be possible, depending on the initial conditions.

2. **Conservation equations with viscosity**

In the presence of a viscous stress tensor, $\boldsymbol{\tau}$, the Navier-Stokes and internal energy equations

[1] Brandenburg, A., Andersen, A. C., Höfner, S., & Nilsson, M., "Homochiral growth through enantiomeric cross-inhibition," *Orig. Life Evol. Biosph.* **35**, 225-241 (2005).

take the form

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{8}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} e + \frac{P}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\tau} \cdot \boldsymbol{\nabla} \boldsymbol{u}, \tag{9}$$

where e is the internal energy per unit mass.

(a) How are the right-hand sides of the inviscid momentum and energy equations to be modified?

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij} P + \ldots\ldots) \tag{10}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j \left(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e + P\right) + \ldots\ldots\right], \tag{11}$$

(b) How does viscosity affect the jump conditions for one-dimensional shocks?

(c) In the diffusion approximation, how does a radiative flux proportional to $\boldsymbol{F}_{\rm rad} = -K\boldsymbol{\nabla} T$ affect the above equations? What are the consequences for the jump conditions?

(d) In the optically thin case, the diffusion approximation does not hold. Speculate in words how this might affect the conclusions above.

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(a) In the presence of viscosity, we have additional terms on the right-hand sides of the momentum and internal energy equations

$$\frac{\partial}{\partial t}(\rho u_i) = \ldots + \tau_{ij,j}$$

$$\frac{\partial}{\partial t}(\rho e) = \ldots + u_{i,j}\tau_{ij}$$

The $\tau_{ij,j}$ term is readily included under the divergence. To compute the evolution of kinetic energy, we multiply by u_i, so

$$u_i \frac{\partial}{\partial t}(\rho u_i) = \ldots + u_i \tau_{ij,j}$$

but

$$u_i \tau_{ij,j} = [u_i \tau_{ij}]_{,j} - u_{i,j}\tau_{ij}.$$

Thus, when computing

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j \left(\tfrac{1}{2}\rho \boldsymbol{u}^2 + \rho e + P\right) - u_i \tau_{ij}\right],$$

the source and sinks terms ($\pm u_{i,j}\tau_{ij}$) have dropped out, so the equations are, again, in conservation form.

(b) Viscosity just smoothes the jump, but it does not affect the differences sufficiently far away from the shock.

(c) The $\boldsymbol{F}_{\rm rad} = -K\boldsymbol{\nabla}T$ is readily included underneath the divergence,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e) = -\frac{\partial}{\partial x_j}\left[u_j\left(\tfrac{1}{2}\rho\boldsymbol{u}^2 + \rho e + P\right) - u_i\tau_{ij} - F_j^{\rm rad}\right],$$

The presence of the $\boldsymbol{F}_{\rm rad} = -K\boldsymbol{\nabla}T$ term does not affect the jump conditions, because far away from the shock the temperature is supposed to be constant.

(d) In the optically thin case, we have a nonlocal loss term

$$\frac{\partial}{\partial t}(\rho e) = ... + \kappa\rho\oint_{4\pi}(I - S)\,\mathrm{d}\Omega,$$

which cannot be written underneath the divergence term. So we can't write the jump conditions in the usual way. (Here, S is the source function, I is the intensity, and κ is the opacity.)

3. **Rayleigh equation from linearized stream function**

In two dimensions, (x, y), the velocity can be written as $\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{z}})$, where ψ is the stream function.

(a) Derive the inviscid, but fully nonlinear evolution equation for $\nabla^2\psi$.

(b) Linearize this evolution equation by writing $\psi(x, y, t) = \psi_0(x) + \psi_1(x, y, t)$.

(c) Define $\boldsymbol{U} = \boldsymbol{\nabla}\times(\psi_0\hat{\boldsymbol{z}})$ as the background flow and thus derive Rayleigh's instability equation for $\psi_1 = \psi_1(x, y, t)$ directly from the equation for the stream function. Assume $\psi_1(x, y, t) = \hat{\psi}_1(x)\,e^{\mathrm{i}k(y-ct)}$, where c is a wave speed.

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(a) We recall that $\omega = -\nabla^2\psi$, so the vorticity equation in 2-D ($\mathrm{D}\omega/\mathrm{D}t = 0$) can be written in the form

$$\frac{\partial}{\partial t}\nabla^2\psi + [\boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{z}})]\cdot\boldsymbol{\nabla}\nabla^2\psi = 0,$$

which can also be written in terms of a Jacobian,

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0.$$

(b) Linearizing this yields

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \frac{\partial(\psi_1, \nabla^2\psi_0)}{\partial(x, y)} + \frac{\partial(\psi_0, \nabla^2\psi_1)}{\partial(x, y)} = 0.$$

(c) Using $U = \partial\psi/\partial x$, we have

$$\frac{\partial}{\partial t}\nabla^2\psi_1 - \frac{\partial\psi_1}{\partial y}U'' + U\frac{\partial\nabla^2}{\partial y} = 0$$

Assuming $\psi_1(x, y, t) = \hat{\psi}_1(x)\,e^{\mathrm{i}k(y-ct)}$, we have

$$\mathrm{i}kc\,k^2\hat{\psi}_1 - \mathrm{i}k\hat{\psi}_1U'' - \mathrm{i}kU\,k^2\hat{\psi}_1 = 0,$$

or

$$(U - c)\,(D^2 - k^2)\hat{\psi}_1 - \hat{\psi}_1U''.$$

4. **Solutions to Rayleigh's instability equation**

Consider Rayleigh's instability equation in the form

$$(U - c)\left(\partial_x^2 - k^2\right)\hat{\psi} - U''\hat{\psi} = 0, \tag{12}$$

rewrite it,

$$\left[U\left(\partial_x^2 - k^2\right) - U''\right]\hat{\psi} = c\left(\partial_x^2 - k^2\right)\hat{\psi} \tag{13}$$

and formulate a numerical eigenvalue problem in the form $\mathsf{M}\boldsymbol{q} = c\mathsf{S}\boldsymbol{q}$, where c is the eigenvalue and $\boldsymbol{q}$ the eigenvector consisting of the discretized values of $\hat{\psi}$. See solutions for eigenvalues and eigenvectors in Fig. 7.

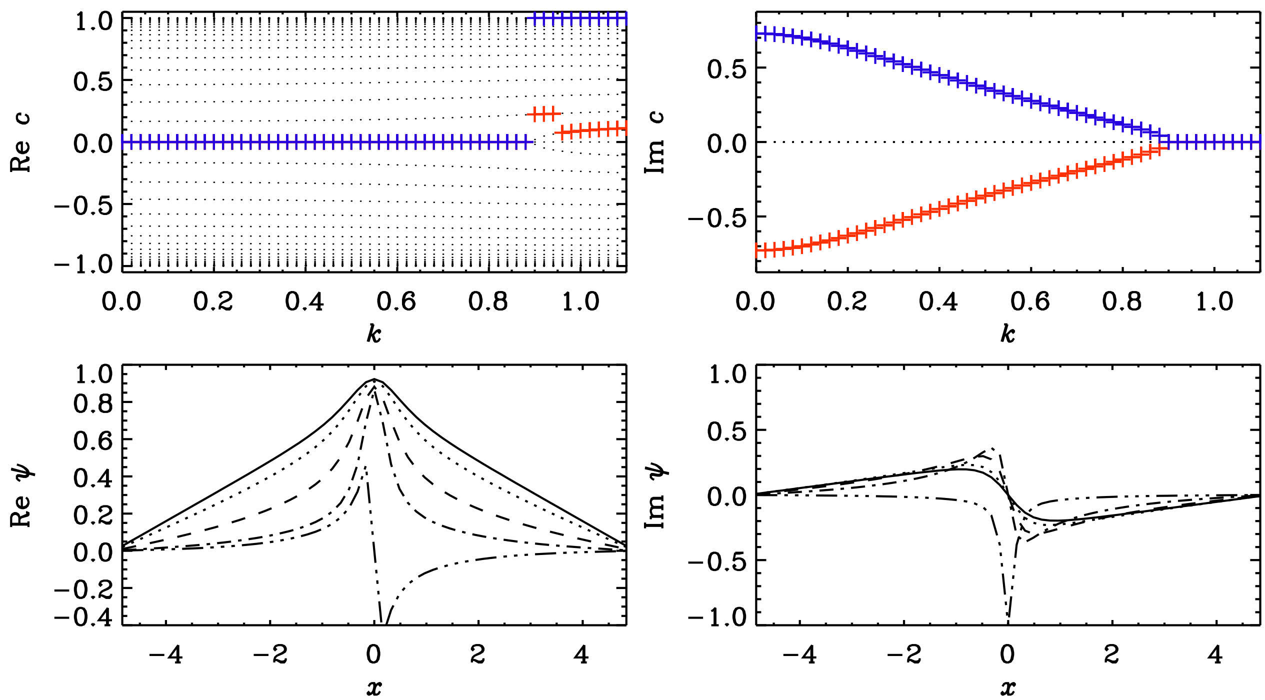

Figure 7: Eigenvalues (upper panels) and eigenvectors (lower panels) for $U = \tanh x$ in $-5 < x < 5$. Eigenvectors are shown for $k = 0$ (solid), 0.2 (dotted), 0.4 (dashed), 0.6 (dash-dotted), and 0.8 (dash-triple-dotted); see `http://lcd-www.colorado.edu/~axbr9098/teach/ASTR_5410/lectures/8_Inflection_Pt_Inst_II/idl/`.

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The details of constructing the matrix were explained during the lecture. `http://lcd-www.colorado.edu/~axbr9098/teach/ASTR_5410/lectures/8_Inflection_Pt_Inst_II/idl/`. Here here visualizations of the matrices, together with explicit numbers.

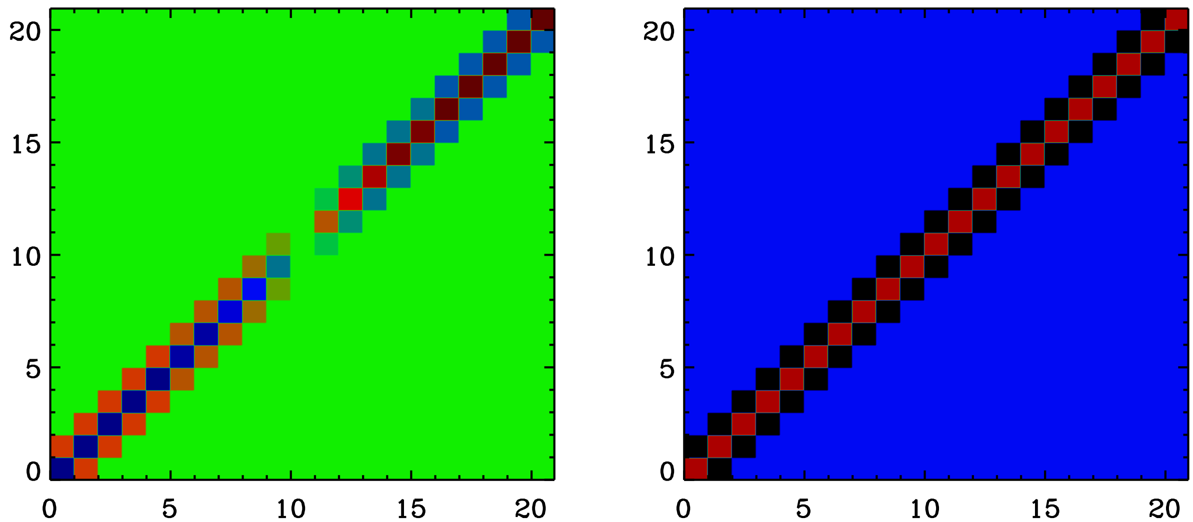
1. 

Figure 8: Visualization of the **M** and **S** matrices.

10.886	-4.837	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
-4.839	10.881	-4.833	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
-0.000	-4.837	10.868	-4.823	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
-0.000	-0.000	-4.833	10.835	-4.799	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
-0.000	-0.000	-0.000	-4.823	10.754	-4.738	-0.000	-0.000	-0.000	-0.000	0.000
-0.000	-0.000	-0.000	-0.000	-4.799	10.559	-4.591	-0.000	-0.000	-0.000	0.000
-0.000	-0.000	-0.000	-0.000	-0.000	-4.738	10.107	-4.246	-0.000	-0.000	0.000
-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-4.591	9.130	-3.488	-0.000	0.000
-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-4.246	7.252	-2.060	0.000
-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-3.488	4.152	0.000
-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-2.060	-0.000

-10.890	4.840	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4.840	-10.890	4.840	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	4.840	-10.890	4.840	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	4.840	-10.890	4.840	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	4.840	-10.890	4.840	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	4.840	-10.890	4.840	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	4.840	-10.890	4.840	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	4.840	-10.890	4.840	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	4.840	-10.890	4.840	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	4.840	-10.890	4.840
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	4.840	-10.890