ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 4 (Due Wed., Nov 2, 2016) November 29, 2016, Axel Brandenburg

1. **Jeans instability with rotation.** In the presence of rotation one has to take the Coriolis force into account. The Euler linearized equations takes then the following form

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1, \tag{1}$$

$$\frac{\partial \boldsymbol{u}_1}{\partial t} = -c_\mathrm{s}^2 \boldsymbol{\nabla} \rho_1 - 2\boldsymbol{\Omega} \times \boldsymbol{u}_1 - \boldsymbol{\nabla} \Phi_1, \tag{2}$$

$$\nabla^2 \Phi_1 = 4\pi G \rho_1, \tag{3}$$

where $\boldsymbol{\Omega} = (0, 0, \Omega)$ is the rotation vector and Ω is the (constant) rotation rate. Subscripts 1 indicate small perturbations, and c_s and ρ_0 are also assumed constant,

(a) Show that the linearized Fourier transformed equations can be written in matrix form

$$M_{ij} q_j = 0 \tag{4}$$

where

$$M_{ij} = \begin{pmatrix} \sigma & \mathrm{i}k_x\rho_0 & \mathrm{i}k_y\rho_0 & \mathrm{i}k_z\rho_0 & 0 \\ \mathrm{i}k_x c^2/\rho_0 & \sigma & -2\Omega & 0 & \mathrm{i}k_x \\ \mathrm{i}k_y c^2/\rho_0 & 2\Omega & \sigma & 0 & \mathrm{i}k_y \\ \mathrm{i}k_z c^2/\rho_0 & 0 & 0 & \sigma & \mathrm{i}k_z \\ 4\pi G & 0 & 0 & 0 & k^2 \end{pmatrix}, \quad q = \begin{pmatrix} \hat{\rho} \\ \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \\ \hat{\Phi} \end{pmatrix}. \tag{5}$$

(b) Find the dispersion relation $\sigma = \sigma(\boldsymbol{k})$ assuming an isothermal equation of state, i.e., $P/\rho = c_\mathrm{s}^2 = \text{const.}$

(c) Refine the problem by considering a small amount of viscosity by writing the momentum equation in the form

$$\frac{\partial \boldsymbol{u}_1}{\partial t} - \nu \nabla^2 \boldsymbol{u}_1 = -c_\mathrm{s}^2 \boldsymbol{\nabla} \rho_1 - 2\boldsymbol{\Omega} \times \boldsymbol{u}_1 - \boldsymbol{\nabla} \Phi_1, \tag{6}$$

How does the presence of ν modify the solution. You may solve the dispersion relation numerically. Plot the results as functions of k_z and $k_\perp$ separately. Here, $k_\perp^2 = k_x^2 + k_y^2$.

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(a) The first row corresponds to the continuity equation. The next 3 correspond to the momentum equation. In here, there are two negative gradient terms (operating on ρ and on ϕ), corresponding to $+\mathrm{i}\boldsymbol{k}$ on the left-hand side. The Coriolis term on the left-hand side gives

$$2\boldsymbol{\Omega} \times \hat{\boldsymbol{u}}_1 = \begin{pmatrix} 0 \\ 0 \\ 2\Omega \end{pmatrix} \times \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix} = \begin{pmatrix} -2\Omega u_y \\ +2\Omega u_x \\ 0 \end{pmatrix}. \tag{7}$$

which explains the -2Ω and $+2\Omega$ terms. Finally we have the $\nabla^2\Phi_1 = 4\pi G \rho_1$ equation, which becomes $k^2\Phi_1 + 4\pi G\rho_1 = 0$ and explains the terms in the last row,

(b) To find nontrivial solutions the determinant of M_{ij} has to vanish. The determinant is given by

$$\boldsymbol{k}^{-2}\det M = \sigma^4 + \sigma^2(c^2\boldsymbol{k}^2 + 4\Omega^2 - 4\pi G\rho_0) + 4\Omega^2 k_z^2(c^2\boldsymbol{k}^2 - 4\pi G\rho_0)/\boldsymbol{k}^2 = 0 \tag{8}$$

This is a quadratic equation in σ^2. Its solution for σ^2 is

$$\sigma_\pm^2 = -\tfrac{1}{2}(c^2\boldsymbol{k}^2 + 4\Omega^2 - 4\pi G\rho_0) \pm \sqrt{\tfrac{1}{4}(c^2\boldsymbol{k}^2 + 4\Omega^2 - 4\pi G\rho_0)^2 - 4\Omega^2 k_z^2(c^2\boldsymbol{k}^2 - 4\pi G\rho_0)/\boldsymbol{k}^2}. \tag{9}$$

Consider limiting cases. (a) $\Omega = 0$, so

$$\sigma_\pm^2 = -\tfrac{1}{2}(c^2\boldsymbol{k}^2 - 4\pi G\rho_0) \pm \sqrt{\tfrac{1}{4}(c^2\boldsymbol{k}^2 - 4\pi G\rho_0)^2}. \tag{10}$$

i.e., $\sigma_+^2 = 0$ and $\sigma_-^2 = -(c^2\boldsymbol{k}^2 - 4\pi G\rho_0)$.

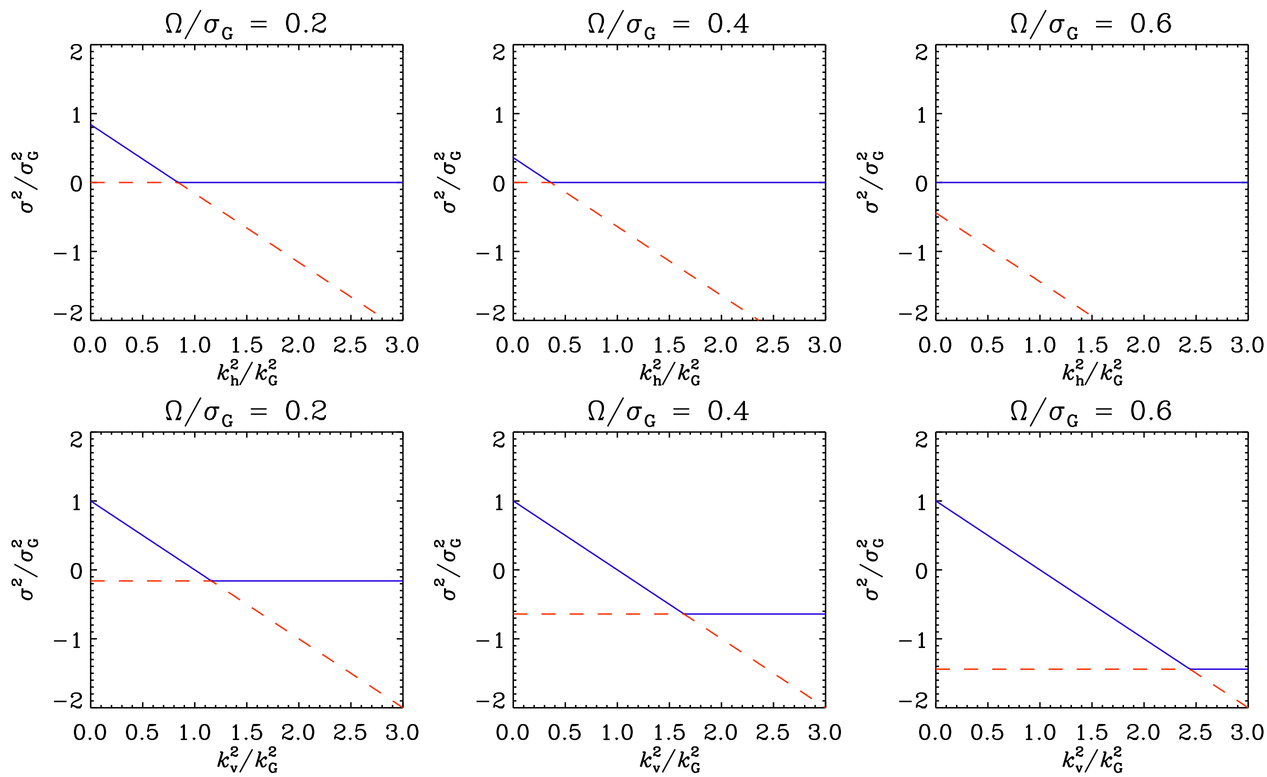

Figure 1: Dispersion relation showing σ^2/σ_0^2 for three values of Ω as a function of either k_h^2 (upper row) or k_v^2 (lower row). Here, $\sigma_0^2 = G\rho_0$ and $k_0 = \sigma_0/c_\mathrm{s}$ have been used for normalization.

In Figure 1 we plot σ^2 in units of $\sigma_0^2 \equiv 4\pi G\rho_0$ as a function of $\boldsymbol{k}^2$ for different values of Ω^2. Interestingly, the function $\sigma(k_\mathrm{v})$ is *not* changed, but instead a new (horizontal) branch appears which is independent of $\boldsymbol{k}$. On the other hand, $\sigma(k_\mathrm{h})$ shifts downward with increasing values of Ω.

(c) Consider now the problem with viscosity and let us introduce $\tilde{\sigma} = \sigma + \nu\boldsymbol{k}^2$, so the matrix takes the form

$$M_{ij} = \begin{pmatrix} \sigma & \mathrm{i}k_x\rho_0 & \mathrm{i}k_y\rho_0 & \mathrm{i}k_z\rho_0 & 0 \\ \mathrm{i}k_x c^2/\rho_0 & \tilde{\sigma} & -2\Omega & 0 & \mathrm{i}k_x \\ \mathrm{i}k_y c^2/\rho_0 & 2\Omega & \tilde{\sigma} & 0 & \mathrm{i}k_y \\ \mathrm{i}k_z c^2/\rho_0 & 0 & 0 & \tilde{\sigma} & \mathrm{i}k_z \\ 4\pi G & 0 & 0 & 0 & \boldsymbol{k}^2 \end{pmatrix}, \quad q = \begin{pmatrix} \hat{\rho} \\ \hat{u}_x \\ \hat{u}_y \\ \hat{u}_z \\ \hat{\Phi} \end{pmatrix}. \tag{11}$$

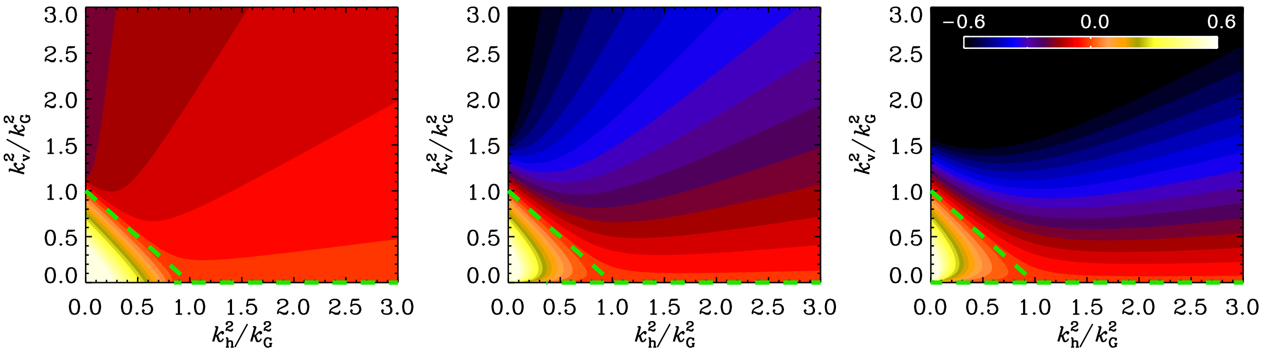

Figure 2: 2-D representation of the dispersion relation for three values of Ω. The green line marks the marginal line for onset.

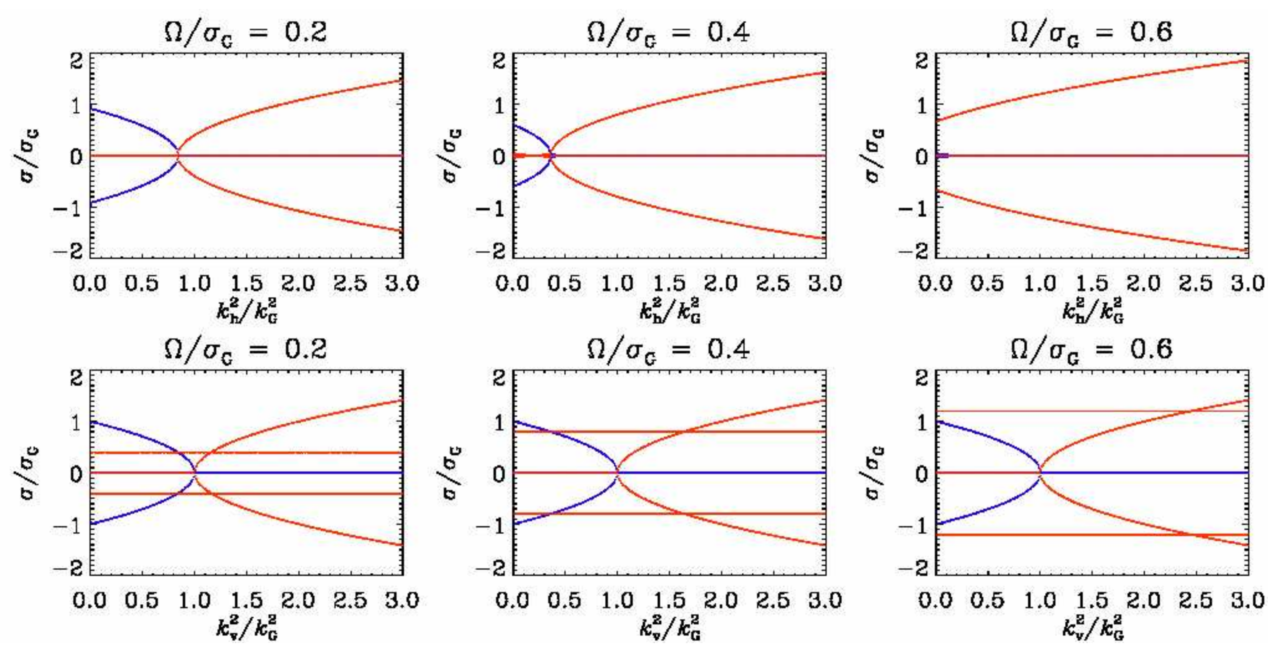

Figure 3: Dispersion relation showing σ/σ_0 (instead of σ^2/σ_0^2) for $\nu = 0$. Blue (red) denotes the real (imaginary) part.

and compute the determinant,

$$\det \mathbf{M} = 4\pi G \left\{-\tilde{\sigma}\left[2\Omega(k_x k_y - kx_k y)\rho_0 + \tilde{\sigma} k_\perp^2 \rho_0\right] + \mathrm{i}k_z\left[\mathrm{i}k_z\rho_0\tilde{\sigma}^2 + 4\Omega^2\mathrm{i}k_z\rho_0\right]\right\} \quad (12)$$

$$+ \ \boldsymbol{k}^2\left\{-(\mathrm{i}k_z c^2/\rho_0)\,\mathrm{i}k_z\rho_0\,(\tilde{\sigma}^2 + 4\Omega^2) + \tilde{\sigma}(\sigma\tilde{\sigma}^2 + \tilde{\sigma}c^2k_\perp^2 + 4\Omega^2\sigma)\right\} \quad (13)$$

Simplify

$$\det \mathbf{M} = 4\pi G\rho_0\left\{-\tilde{\sigma}^2 k_\perp^2 - k_z^2\left(\tilde{\sigma}^2 + 4\Omega^2\right)\right\} \quad (14)$$

$$+ \ \boldsymbol{k}^2\left\{c^2k_z^2\,(\tilde{\sigma}^2 + 4\Omega^2) + (\sigma\tilde{\sigma}^3 + \tilde{\sigma}^2c^2k_\perp^2 + 4\Omega^2\sigma\tilde{\sigma})\right\} \quad (15)$$

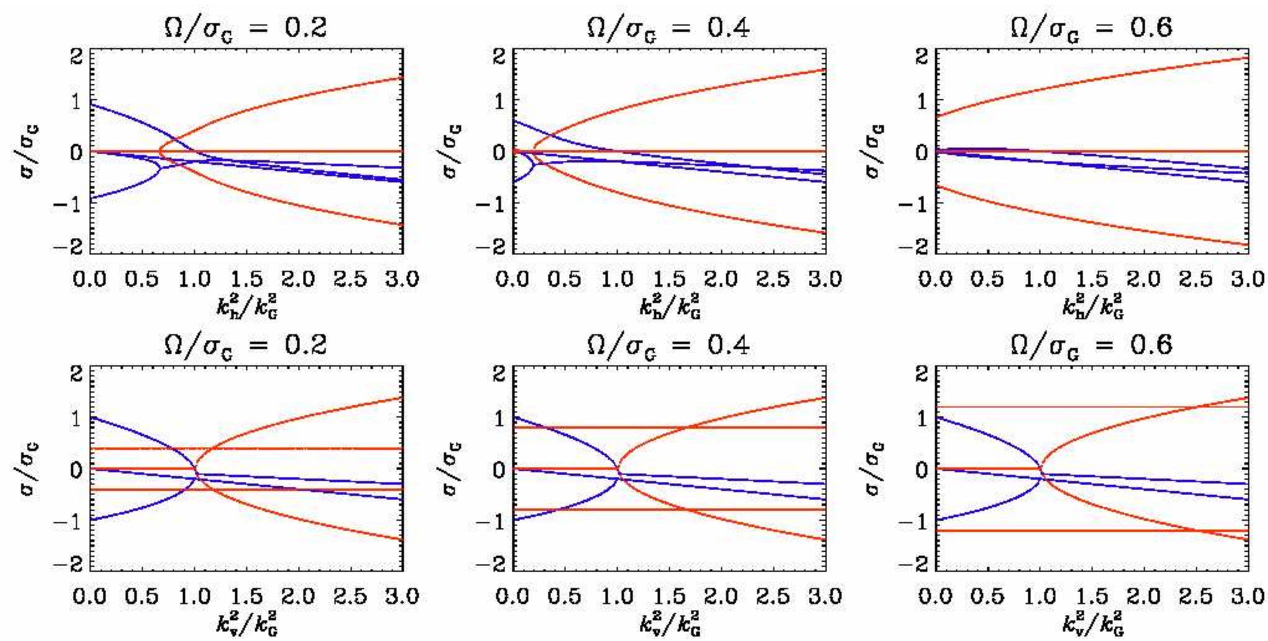

Figure 4: Dispersion relation showing σ/σ_0 for $\nu = 0.2$.

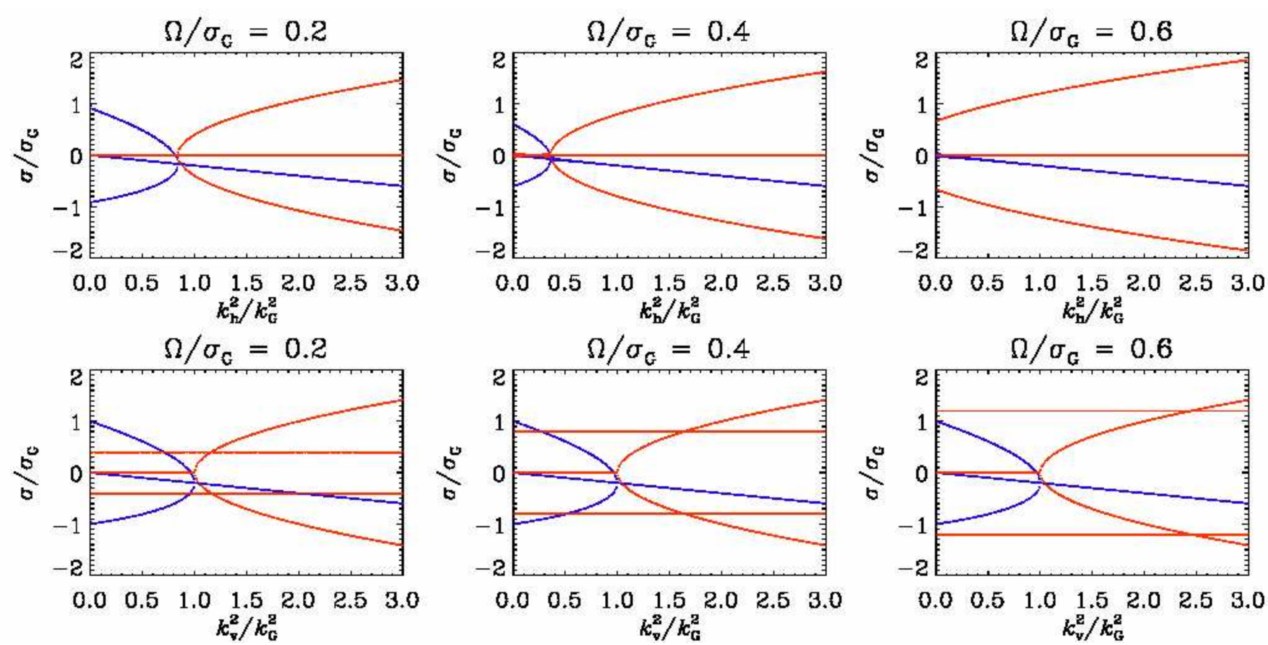

Figure 5: Dispersion relation showing σ/σ_0 for $\nu = \mu = 0.2$, where μ is an *artificial* diffusion coefficient in the continuity equation. Comparing with Figure 4, we see certain blue lines now collapsing on top of each other.

or, dividing by $\boldsymbol{k}^2$ and introducing $\sigma_G^2 \equiv 4\pi G\rho_0$,

$$k^{-2}\det\mathsf{M} = -\sigma_G^2\left(\tilde{\sigma}^2 + 4\Omega^2 k_z^2/\boldsymbol{k}^2\right) + c^2k_z^2\left(\tilde{\sigma}^2 + 4\Omega^2\right) + \sigma\tilde{\sigma}^3 + \tilde{\sigma}^2c^2k_\perp^2 + 4\Omega^2\sigma\tilde{\sigma}. \quad (16)$$

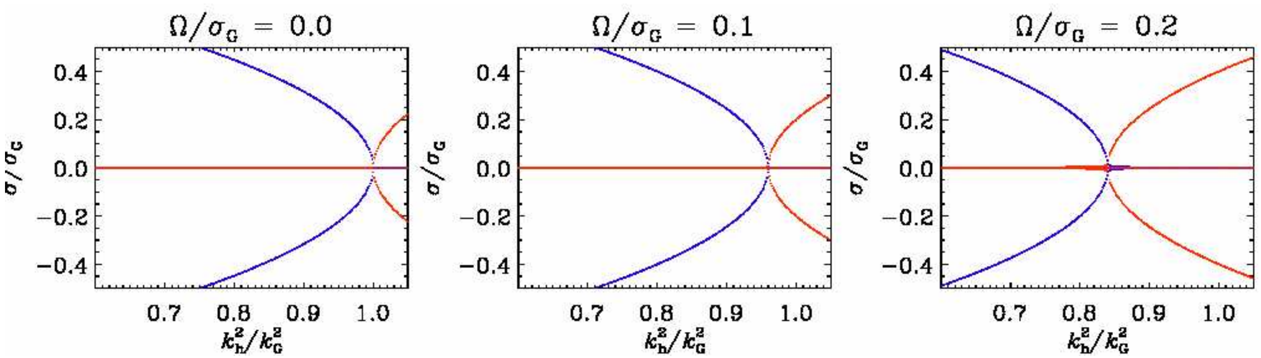

Figure 6: Zoom-in near onset; $\nu = 0$. Otherwise like upper row of Figure 3.

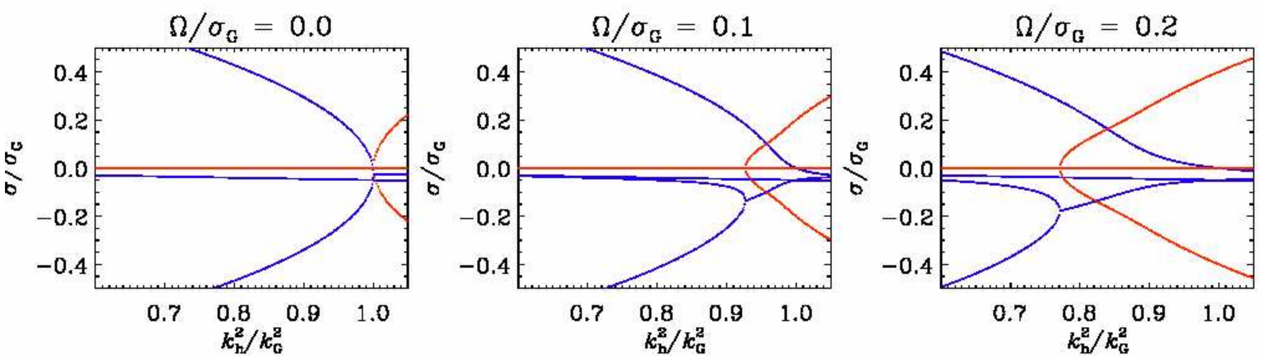

Figure 7: Zoom-in near onset; $\nu = 0.05$. Note that for finite rotation, viscosity extends to range of instability.

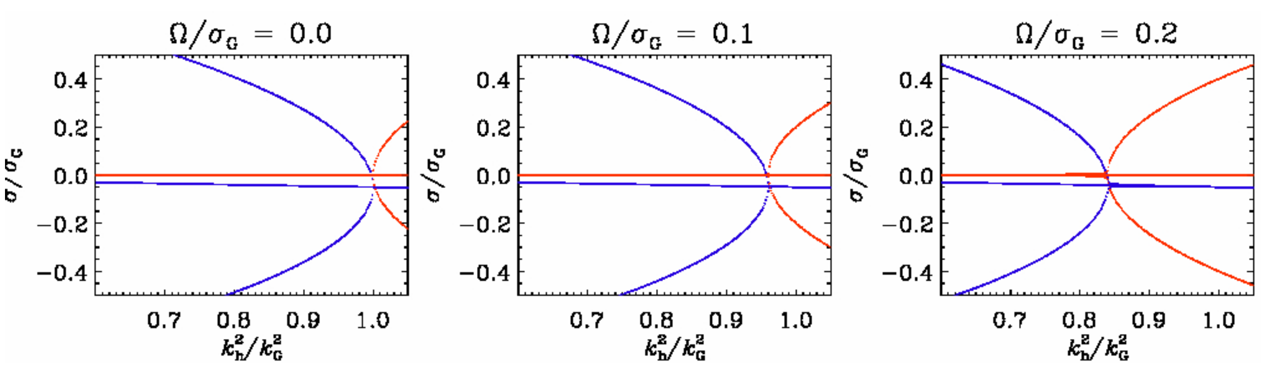

Figure 8: Similar to Figure 7, but now also with $\mu = 0.05$. The extended range of instability is *not* reproduced with this trick!

Setting now det M to zero and simplifying further, we have

$$-\tilde{\sigma}^2\sigma_G^2 + 4\Omega^2(c^2\boldsymbol{k}^2 - \sigma_G^2)\,(k_z^2/\boldsymbol{k}^2) + \sigma\tilde{\sigma}^3 + \tilde{\sigma}^2 c^2\boldsymbol{k}^2 + 4\Omega^2\sigma\tilde{\sigma} = 0. \tag{17}$$

Reordering this gives

$$\sigma\tilde{\sigma}^3 + \tilde{\sigma}^2(c^2\boldsymbol{k}^2 - \sigma_G^2) + 4\Omega^2\sigma\tilde{\sigma} + 4\Omega^2(c^2\boldsymbol{k}^2 - \sigma_G^2)\,(k_z^2/\boldsymbol{k}^2) = 0, \tag{18}$$

so we see that there are only two mixed terms. We also see that it readily reduces to the simpler form when $\nu = 0$. Next, let us introduce $A = c^2\boldsymbol{k}^2 - \sigma_G^2$, $B = 4\Omega^2$, and $C = 4\Omega^2(c^2\boldsymbol{k}^2 - \sigma_G^2)\,(k_z^2/\boldsymbol{k}^2)$, we have

$$\sigma\tilde{\sigma}^3 + \tilde{\sigma}^2 A + \sigma\tilde{\sigma} B + C = 0. \tag{19}$$

In Figure 4 we show a numerical solution obtained by solving the 5×5 matrix eigenvalue problem numerically. The solution turns out to be close to one obtained by adding an *artificial* diffusion coefficient in the continuity equation, which allows for an analytical solution,

$$\sigma_{\pm}^{\pm} = -\nu k^2 \pm \left\{-\tfrac{1}{2}(c^2\boldsymbol{k}^2 + 4\Omega^2 - 4\pi G\rho_0) \pm \sqrt{\tfrac{1}{4}(c^2\boldsymbol{k}^2 + 4\Omega^2 - 4\pi G\rho_0)^2 - 4\Omega^2 k_z^2(c^2\boldsymbol{k}^2 - 4\pi G\rho_0)/\boldsymbol{k}^2}\right\}^{1/2}. \tag{20}$$

The result is shown in Figure 5.

2. **Dimensional arguments.** Use dimensional arguments to determine the form of the energy spectrum $E(k)$.

   (a) In two-dimensional turbulence the rate of enstrophy injection, $\beta = \frac{d}{dt}\langle\boldsymbol{\omega}^2\rangle$, is constant and independent of wavenumber k. Use dimensional arguments to find the energy spectrum $E(k)$. [Hints: the energy spectrum is normalized so that $\int_0^\infty E(k)\,dk = \frac{1}{2}\langle\mathbf{u}^2\rangle$ and $E(k)$ depends only β and k.]

   (b) Now consider hydromagnetic turbulence and assume that the spectrum can be written in the form

   $$E(k) = C\,(v_{\rm A}\epsilon)^a k^b,$$

   where C is a dimensionless constant, $v_{\rm A}$ is the Alfvén speed, ϵ (with dimension $\mathrm{m^2\,s^{-3}}$) the energy injection rate, and k the wavenumber. [Note that $\int E(k)dk$ has the dimension $\mathrm{m^2\,s^{-2}}$.]

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(a) Make the ansatz

$$E(k) = C\beta^a k^b, \tag{21}$$

use the facts that $[E(k)] = \mathrm{m^3\,s^{-2}}$, $[\beta] = \mathrm{s^{-3}}$, and $[k] = \mathrm{m^{-1}}$, and balance the equation for the dimensions m and s like so

$$\mathrm{m}: \qquad +3 = +0a - 1b \tag{22}$$

$$\mathrm{s}: \qquad -2 = -3a - 0b \tag{23}$$

so we have 2 equations with the two unknowns a and b. The solution is here particularly simple, so

$$b = -3, \quad a = 2/3. \tag{24}$$

Inserting this into Equation (21) yields

$$E(k) = C\beta^{2/3}k^{-3}. \tag{25}$$

Thus, we have a k^{-3} spectrum in 2-D.

(b) Again, make the ansatz

$$E(k) = C(v_A\epsilon)^a k^b, \tag{26}$$

use the facts that $[E(k)] = \mathrm{m^3\,s^{-2}}$, $[v_A\epsilon] = \mathrm{m^3\,s^{-4}}$, and $[k] = \mathrm{m^{-1}}$, and balance the equation for the dimensions m and s, so

$$\mathrm{m}: \quad +3 = +3a - 1b \tag{27}$$
$$\mathrm{s}: \quad -2 = -4a - 0b \tag{28}$$

The solution is now

$$a = 1/2, \quad b = 3a - 3 = 3/2 - 3 = -3/2. \tag{29}$$

Inserting this into Equation (26) yields

$$E(k) = C(v_A\epsilon)^{1/2} k^{-3/2}. \tag{30}$$

Thus, we expect a $k^{-3/2}$ spectrum in MHD. This is called the Iroshnikov[1]–Kraichnan[2] spectrum. It is nowadays superseded by the Goldreich–Sridhar[3] spectrum.

3. **Hydrodynamic turbulence.**

Take the kinetic energy spectrum of hydrodynamic turbulence to be of the form

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}, \tag{31}$$

where C_K is the Kolmogorov constant.

(a) To calculate the length of the inertial range, assume that $E(k)$ is finite only in the range $k_f \le k \le k_d$. Thus, $u_{\rm rms}$ and ϵ are given by the two integrals

$$\tfrac{1}{2}u_{\rm rms}^2 = \int_{k_f}^{k_d} E(k)\,\mathrm{d}k \approx \tfrac{3}{2} C_K \epsilon^{2/3} k_f^{-2/3}, \tag{32}$$

$$\epsilon = \int_{k_f}^{k_d} 2\nu k^2 E(k)\,\mathrm{d}k \approx \tfrac{3}{2}\nu C_K \epsilon^{2/3} k_d^{4/3}, \tag{33}$$

which are just the normalization condition of $E(k)$ and the definition of the energy dissipation, respectively. Here, ν is the kinematic viscosity.

(b) Eliminating ϵ, and writing the result in terms of the Reynolds number, show that

$$\mathrm{Re} = \frac{u_{\rm rms}}{\nu k_f} \approx \tfrac{3}{2}\sqrt{3}\, C_K^{3/2} \left(\frac{k_d}{k_f}\right)^{4/3}. \tag{34}$$

Thus, the length of the inertial range scales with the Reynolds number like $k_d/k_f \approx \mathrm{Re}^{3/4}$

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[1] Iroshnikov, R. S., "Turbulence of a conducting fluid in a strong magnetic field," *Sov. Astron.* **7**, 566-571 (1964).
[2] Kraichnan, R. H., "Inertial-range spectrum of hydromagnetic turbulence," *Phys. Fluids* **8**, 1385-1387 (1965).
[3] Goldreich, P., & Sridhar, S., "Toward a theory of interstellar turbulence," *Astrophys. J.* **438**, 763-775 (1995).

(a) Using $E(k) = C_{\rm K}\epsilon^{2/3}k^{-5/3}$, we compute the integral over $E(k)$ in the range $k_{\rm f} \le k \le k_{\rm d}$ and obtain

$$\tfrac{1}{2}u_{\rm rms}^2 = \int_{k_{\rm f}}^{k_{\rm d}} E(k)\,{\rm d}k = \int_{k_{\rm f}}^{k_{\rm d}} C_{\rm K}\epsilon^{2/3}k^{-5/3}\,{\rm d}k = -\,\tfrac{3}{2}C_{\rm K}\epsilon^{2/3}k^{-2/3}\Big|_{k_{\rm f}}^{k_{\rm d}} \tag{35}$$

which gives

$$\tfrac{1}{2}u_{\rm rms}^2 = -\tfrac{3}{2}C_{\rm K}\epsilon^{2/3}k_{\rm d}^{-2/3} + \tfrac{3}{2}C_{\rm K}\epsilon^{2/3}k_{\rm f}^{-2/3} \approx \tfrac{3}{2}C_{\rm K}\epsilon^{2/3}k_{\rm f}^{-2/3}, \tag{36}$$

because the first term is small ($k_{\rm d}$ is large, but the exponent negative). Next, we compute

$$\epsilon = \int_{k_{\rm f}}^{k_{\rm d}} 2\nu k^2 E(k)\,{\rm d}k = \int_{k_{\rm f}}^{k_{\rm d}} 2\nu\, C_{\rm K}\epsilon^{2/3}k^{1/3}\,{\rm d}k = 2\nu\,\tfrac{3}{4}C_{\rm K}\epsilon^{2/3}k^{4/3}\Big|_{k_{\rm f}}^{k_{\rm d}}, \tag{37}$$

so

$$\epsilon = \tfrac{3}{2}\nu\, C_{\rm K}\epsilon^{2/3}k_{\rm d}^{4/3} - \tfrac{3}{2}\nu\, C_{\rm K}\epsilon^{2/3}k_{\rm f}^{4/3} \approx \tfrac{3}{2}\nu C_{\rm K}\epsilon^{2/3}k_{\rm d}^{4/3}, \tag{38}$$

where we could this time skip the second part ($k_{\rm f}$ is small by comparison and the exponent positive).

(b) In Equation (38), ϵ appears twice, so we first have

$$\epsilon^{1/3} = \tfrac{3}{2}\nu C_{\rm K}k_{\rm d}^{4/3}. \tag{39}$$

From Equation (36) we have

$$u_{\rm rms} = \sqrt{3C_{\rm K}}\epsilon^{1/3}k_{\rm f}^{-1/3}, \tag{40}$$

Inserting the expression for $\epsilon^{1/3}$ yields

$$u_{\rm rms} = \sqrt{3C_{\rm K}}\,\tfrac{3}{2}\nu C_{\rm K}k_{\rm d}^{4/3}k_{\rm f}^{-1/3}, \tag{41}$$

so

$$\frac{u_{\rm rms}}{\nu k_{\rm f}} = \tfrac{3}{2}\sqrt{3}\,C_{\rm K}^{3/2}k_{\rm d}^{4/3}k_{\rm f}^{-4/3}, \tag{42}$$

which agrees with Equation (34). So the length of the inertial range is $k_{\rm d}/k_{\rm f} \approx {\rm Re}^{3/4}$.

4. **The alpha effect.** Consider the following evolution equation for the mean electromotive force $\boldsymbol{\mathcal{E}}$ in the form

$$\boldsymbol{\mathcal{E}} = \tau\overline{\boldsymbol{u}\times\boldsymbol{\nabla}\times(\boldsymbol{u}\times\overline{\boldsymbol{B}})} \tag{43}$$

(a) Show that

$$\mathcal{E}_i = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\,\tau\overline{u_j\partial_l u_n\overline{B}_p} \tag{44}$$

(b) Assume that $\overline{\boldsymbol{B}}$ can be pulled outside the averages and define rank 2 and rank 3 tensors such that

$$\mathcal{E}_i = \alpha_{ip}\overline{B}_p + \eta_{ipl}\overline{B}_{p,l} \tag{45}$$

(c) Show that

$$\alpha_{ip} = \epsilon_{jnp}\,\tau\overline{u_j u_{n,i}} - \epsilon_{inp}\,\tau\overline{u_j u_{n,j}} \tag{46}$$

(d) Assume isotropy and define $\alpha = \frac{1}{3}\delta_{ip}\alpha_{ip}$. Show that

$$\alpha = -\tfrac{1}{3}\tau\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}. \tag{47}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$.

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(a) Consider the ith component and replace cross products by epsilons, so

$$\mathcal{E}_i = \tau\overline{\epsilon_{ijk}u_j\epsilon_{klm}\partial_l\epsilon_{mnp}\, u_n\overline{B}_p} = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\,\tau\overline{u_j\partial_l u_n\overline{B}_p}. \tag{48}$$

Here, in the last step, we have just moved all the epsilons to the left.

(b) Writing

$$\overline{u_j\partial_l u_n\overline{B}_p} = \overline{u_j u_{n,l}\overline{B}_p} + \overline{u_j u_n\overline{B}_{p,l}} = \overline{u_j u_{n,l}}\,\overline{B}_p + \overline{u_j u_n}\,\overline{B}_{p,l} \tag{49}$$

where we have assumed that the product of averages is equal to the average of products.[4]

(c) The full part before the $\overline{B}_p$ term is

$$\alpha_{ip} = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\,\tau\overline{u_j u_{n,l}}. \tag{52}$$

For the first two epsilons, we write $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, so

$$\alpha_{ip} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\epsilon_{mnp}\,\tau\overline{u_j u_{n,l}} = \epsilon_{jnp}\,\tau\overline{u_j u_{n,i}} - \epsilon_{inp}\,\tau\overline{u_j u_{n,j}}. \tag{53}$$

(d) Isotropy means that α_{ip} is proportional to an *isotropic tensor*. The only isotropic tensor of rank 2 is δ_{ip}, soo we assume $\alpha_{ip} = \alpha\delta_{ip}$. To compute α, we take the trace, i.e., we compute $\alpha = \frac{1}{3}\delta_{ip}\alpha_{ip}$ and replace p by i, so

$$\alpha = \epsilon_{jni}\,\tau\overline{u_j u_{n,i}} - \epsilon_{ini}\,\tau\overline{u_j u_{n,j}}. \tag{54}$$

Here, the last term vanishes, $\epsilon_{ini} = 0$, because two indices are the same. Thus, we are left with

$$\alpha = \tfrac{1}{3}\epsilon_{jni}\,\tau\overline{u_j u_{n,i}} = \tfrac{1}{3}\tau\overline{u_j\epsilon_{jni}\,u_{n,i}} = -\tfrac{1}{3}\tau\overline{u_j\epsilon_{jin}\,\partial_i u_n} = -\tfrac{1}{3}\tau\overline{u_j\omega_j}, \tag{55}$$

and thus $\alpha = -\frac{1}{3}\tau\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}$. This shows that the (famous) α effect usually has to do with kinetic helicity. This is, however, only true in the limit when the microphysical magnetic diffusivity is negligible. If this is not the case, we essentially have $\tau \to (\eta k^2)^{-1}$ and the k^{-2} leads to $k^{-2}\boldsymbol{\omega} \to \boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is the vectorial stream function, so $\boldsymbol{u} = \boldsymbol{\nabla}\times\boldsymbol{\psi}$. This is analogous to the magnetic vector potential; see the book by Krause & Rädler[5] for the full story.

Incidently, the isotropic version of the magnetic diffusivity tensor is called the *turbulent* magnetic diffusivity[6] and is given by $\eta_\mathrm{t} = \frac{1}{3}\tau\overline{\boldsymbol{u}^2}$. When the microphysical magnetic diffusivity is *not* negligible, this becomes $\eta_\mathrm{t} = \frac{1}{3}(\tau/\eta)\,(\overline{\boldsymbol{\psi}^2 - \hat{\boldsymbol{\phi}}^2})$, where $\hat{\boldsymbol{\phi}}$ is the scalar potential of the velocity in the compressible case, which can lead to negative magnetic diffusivities; see Rädler et al.[7]

[4]This is known as one of the Reynolds rules. The full set of Reynolds rules is

$$\overline{\overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{U}}_2} = \overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{U}}_2, \quad \overline{\overline{\boldsymbol{U}}} = \overline{\boldsymbol{U}}, \quad \overline{\overline{\boldsymbol{U}}\boldsymbol{u}} = 0, \quad \overline{\overline{\boldsymbol{U}}_1\,\overline{\boldsymbol{U}}_2} = \overline{\boldsymbol{U}}_1\,\overline{\boldsymbol{U}}_2, \tag{50}$$

$$\overline{\partial\boldsymbol{U}/\partial t} = \partial\overline{\boldsymbol{U}}/\partial t, \quad \overline{\partial\boldsymbol{U}/\partial x_i} = \partial\overline{\boldsymbol{U}}/\partial x_i. \tag{51}$$

Some of these properties are not shared by several other averages; for gaussian filtering $\overline{\overline{\boldsymbol{U}}} \neq \overline{\boldsymbol{U}}$, and for spectral filtering $\overline{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}} \neq \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}$, for example. Note that $\overline{\overline{\boldsymbol{U}}} = \overline{\boldsymbol{U}}$ implies that $\overline{\boldsymbol{u}} = 0$.

[5]Krause, F., & Rädler, K.-H. *Mean-field Magnetohydrodynamics and Dynamo Theory*. Oxford: Pergamon (1980).

[6]Analogously to α, one assumes that η_{ipl} in Equation (45) is proportional to an isotropic tensor of rank 3, and the only one is ϵ_{ipl}, so $\eta_{ipl} = \eta_\mathrm{t}\epsilon_{ipl}$. To extract η_t, one computes $\eta_\mathrm{t} = \frac{1}{6}\epsilon_{ipl}\eta_{ipl}$.

[7]Rädler, K.-H., Brandenburg, A., Del Sordo, F., & Rheinhardt, M., "Mean-field diffusivities in passive scalar and magnetic transport in irrotational flows," *Phys. Rev. E* **84**, 4 (2011).

5. **The alpha squared dynamo.** Consider the averaged induction equation in the form

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} + \eta \nabla^2 \overline{\boldsymbol{B}} \tag{56}$$

and assume $\boldsymbol{\mathcal{E}} = \alpha \overline{\boldsymbol{B}}$ [see part (b) of previous question].

(a) Assume $\alpha = \text{const}$ and assume solutions of the form $\overline{\boldsymbol{B}} = \boldsymbol{B}_0\, e^{\mathrm{i}kz+\sigma t}$ to show that

$$\sigma = -\eta k^2 \pm \alpha k \tag{57}$$

(b) Consider the periodic domain $-\pi < z < \pi$ and find the critical value of α above which self-excited solutions are possible. Express α in terms of a non-dimensional parameter.

(c) Find the value of α (and its non-dimensional counterpart) for which σ is maximized.

(d) *Bonus question:* find the eigenvectors $\boldsymbol{B}_0$.

. . . .

(a) For $\alpha = \text{const}$ (and of course $\eta = \text{const}$) we have a PDE with constant coefficients and can write $\overline{\boldsymbol{B}} = \boldsymbol{B}_0\, e^{\mathrm{i}kz+\sigma t}$. We write this in matrix form, so

$$\sigma \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} = \begin{pmatrix} -\eta k^2 & -\mathrm{i}\alpha k_z & \mathrm{i}\alpha k_y \\ \mathrm{i}\alpha k_z & -\eta k^2 & -\mathrm{i}\alpha k_x \\ -\mathrm{i}\alpha k_y & \mathrm{i}\alpha k_x & -\eta k^2 \end{pmatrix} \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix}. \tag{58}$$

This is a standard eigenvalue problem. The dispersion relation follows from

$$\det \begin{pmatrix} \sigma + \eta k^2 & \mathrm{i}\alpha k_z & -\mathrm{i}\alpha k_y \\ -\mathrm{i}\alpha k_z & \sigma + \eta k^2 & \mathrm{i}\alpha k_x \\ +\mathrm{i}\alpha k_y & -\mathrm{i}\alpha k_x & \sigma + \eta k^2 \end{pmatrix} = 0 \tag{59}$$

and leads to

$$(\sigma + \eta k^2)\left[(\sigma + \eta k^2)^2 - \alpha^2 k^2\right] = 0, \tag{60}$$

with the three solutions

$$\sigma_0 = -\eta k^2, \quad \sigma_\pm = -\eta k^2 \pm |\alpha k|. \tag{61}$$

The eigenfunction corresponding to the eigenvalue $\sigma_0 = -\eta k^2$ is proportional to $\boldsymbol{k}$, but this solution is incompatible with solenoidality and has to be dropped. The two remaining branches are shown in Figure 9.

(b) Marginally excited solutions $\mathrm{Re}\sigma = 0$. In this case, σ is real, and so we have

$$0 = -\eta k_{\mathrm{crit}}^2 \pm |\alpha k_{\mathrm{crit}}|. \tag{62}$$

Here, $k_{\mathrm{crit}} = 0$ is one solution, but it requires and infinitely large domain. For $k_{\mathrm{crit}} \neq 0$, we have

$$k_{\mathrm{crit}} = |\alpha|/\eta. \tag{63}$$

We have self-excited solutions for $k < k_{\mathrm{crit}}$. In a periodic domain $-\pi < z < \pi$, the largest scale corresponds to the wavenumber $k = 1$, so the critical value of α above which self-excited solutions are possible is given by

$$\alpha_{\mathrm{crit}} = \eta k = \eta \quad \text{for } k = 1. \tag{64}$$

A non-dimensional parameter for α is

$$C_\alpha = \alpha/\eta k, \tag{65}$$

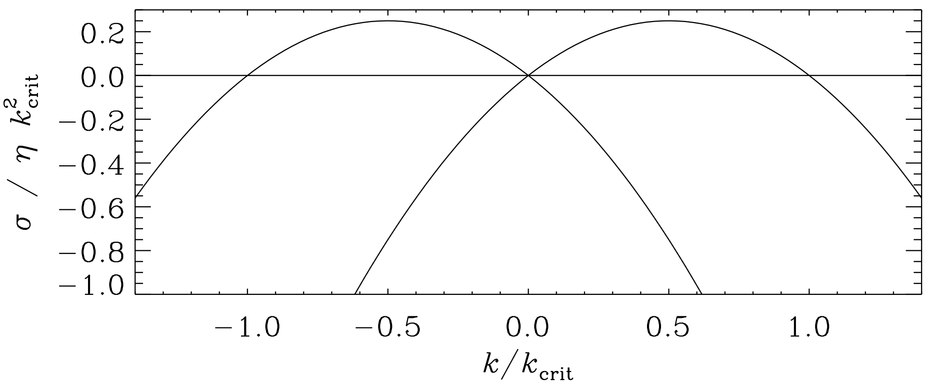

Figure 9: Dispersion relation for α^2 dynamo, where $k_{\rm crit} = \alpha/\eta_{\rm T}$.

(c) The $\sigma(k)$ curve is an inverted parabola, so the maximum is given by $\mathrm{d}\sigma/\mathrm{d}k = 0$, so we have maximum growth at

$$k = k_{\rm max} = k_{\rm crit}/2. \tag{66}$$

Here we have

$$\max(\sigma) = \alpha^2/4\eta. \tag{67}$$

(d) *Bonus question:* To find the eigenvectors $\boldsymbol{B}_0$, we consider the marginally excited state with

$$\sigma = \sigma_+ = -\eta k^2 + \alpha k. \tag{68}$$

and assume positive α and k, so

$$\begin{pmatrix} \alpha k & \mathrm{i}\alpha k_z & -\mathrm{i}\alpha k_y \\ -\mathrm{i}\alpha k_z & \alpha k & \mathrm{i}\alpha k_x \\ +\mathrm{i}\alpha k_y & -\mathrm{i}\alpha k_x & \alpha k \end{pmatrix} \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} = 0. \tag{69}$$

We can normalize the eigenvector such that $B_{0x} = 1$, for example, so we have

$$\begin{pmatrix} \alpha k & \mathrm{i}\alpha k_z & -\mathrm{i}\alpha k_y \\ -\mathrm{i}\alpha k_z & \alpha k & \mathrm{i}\alpha k_x \\ +\mathrm{i}\alpha k_y & -\mathrm{i}\alpha k_x & \alpha k \end{pmatrix} \begin{pmatrix} 1 \\ B_{0y} \\ B_{0z} \end{pmatrix} = 0. \tag{70}$$

so we have 2 unknowns for the remaining two equations, so in matrix form we have

$$\begin{pmatrix} \alpha k & \mathrm{i}\alpha k_x \\ -\mathrm{i}\alpha k_x & \alpha k \end{pmatrix} \begin{pmatrix} B_{0y} \\ B_{0z} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\alpha k_z \\ -\mathrm{i}\alpha k_y \end{pmatrix}. \tag{71}$$

The αs cancel and we are left with $kB_{0y} = \mathrm{i}(k_z - k_x B_{0z})$ from the first row and $kB_{0z} = \mathrm{i}(k_x B_{0y} - k_y)$ from the second, so

$$k^2 B_{0y} = \mathrm{i}(k_z k - k_x k B_{0z}) = \mathrm{i}k_z k + k_x(k_x B_{0y} - k_y), \tag{72}$$

so we have

$$B_{0y} = \frac{\mathrm{i}k_z k - k_x k_y}{k^2 - k_x^2}, \tag{73}$$

and

$$kB_{0z} = \mathrm{i}k_x \frac{\mathrm{i}k_z k - k_x k_y}{k^2 - k_x^2} - \mathrm{i}k_y = \frac{-k_x k_z k - \mathrm{i}k_x^2 k_y}{k^2 - k_x^2} - \mathrm{i}k_y = \frac{-k_x k_z k - \mathrm{i}k_x^2 k_y - \mathrm{i}k_y(k^2 - k_x^2)}{k^2 - k_x^2} \tag{74}$$

so

$$kB_{0z} = \frac{-k_x k_z k - \mathrm{i}k_y k^2}{k^2 - k_x^2} \quad \text{or} \quad B_{0z} = \frac{-k_x k_z - \mathrm{i}k_y k}{k^2 - k_x^2}. \tag{75}$$

Renormalizing the eigenvector once more, we have

$$\begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} \to \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} = \begin{pmatrix} k^2 - k_x^2 \\ \mathrm{i}k_z k - k_x k_y \\ -k_x k_z - \mathrm{i}k_y k \end{pmatrix} = \tag{76}$$

This must then still satisfy the first row of Equation (69), i.e., $kB_{0x} + \mathrm{i}k_z B_{0y} - \mathrm{i}k_y B_{0z} = 0$, so let's check:

$$k(k^2 - k_x^2) + \mathrm{i}k_z(\mathrm{i}k_z k - k_x k_y) - \mathrm{i}k_y(-k_x k_z - \mathrm{i}k_y k) = \tag{77}$$

$$= k(k^2 - k_x^2) - k_z^2 k - \mathrm{i}k_x k_y k_z + \mathrm{i}k_y k_x k_z - k_y^2 k = k(k^2 - k_x^2) - k_z^2 k - k_y^2 k = 0, \tag{78}$$

because $k_x^2 + k_y^2 + k_z^2 = k^2$. Moreover,

$$\begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} \overset{\text{if } k=k_z}{=} \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix} k_z^2 \quad \text{or} \quad \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} \overset{\text{if } k=k_y}{=} \begin{pmatrix} 1 \\ 0 \\ -\mathrm{i} \end{pmatrix} k_y^2. \tag{79}$$

For $k = k_x$ there is a problem and we get zero. The general solution must be an eigenvector of the curl operator, which I found once in an old paper of mine[8], which I gave in the form

$$\frac{\boldsymbol{k} \times (\boldsymbol{k} \times \hat{\boldsymbol{e}}) - i|\boldsymbol{k}|(\boldsymbol{k} \times \hat{\boldsymbol{e}})}{2\boldsymbol{k}^2\sqrt{1 - (\boldsymbol{k} \cdot \hat{\boldsymbol{e}})^2/\boldsymbol{k}^2}}. \tag{80}$$

Thus, we have

$$\begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} \to \begin{pmatrix} B_{0x} \\ B_{0y} \\ B_{0z} \end{pmatrix} = \begin{pmatrix} k_x(k_y + k_z) - (k_y^2 + k_z^2) - \mathrm{i}k(k_y - k_z) \\ k_y(k_z + k_x) - (k_z^2 + k_x^2) - \mathrm{i}k(k_z - k_x) \\ k_z(k_x + k_y) - (k_x^2 + k_y^2) - \mathrm{i}k(k_x - k_y) \end{pmatrix} \tag{81}$$

Test:

$$\begin{gathered} kk_x(k_y + k_z) - k(k_y^2 + k_z^2) - \mathrm{i}k^2(k_y - k_z) \\ +\mathrm{i}k_z k_y(k_z + k_x) - \mathrm{i}k_z(k_z^2 + k_x^2) + kk_z(k_z - k_x) \\ -\mathrm{i}k_y k_z(k_x + k_y) + \mathrm{i}k_y(k_x^2 + k_y^2) - kk_y(k_x - k_y) \end{gathered} \tag{82}$$

One sees quickly that the real terms cancel. For the imaginary terms it looks more messy, so let's write them down separately:

$$-k^2(k_y - k_z) + k_z k_y(k_z + k_x) - k_z(k_z^2 + k_x^2) - k_y k_z(k_x + k_y) + k_y(k_x^2 + k_y^2) \tag{83}$$

There are first two fully mixed terms, $k_x k_y k_z$, that cancel, so

$$-k^2(k_y - k_z) + k_z k_y k_z - k_z(k_z^2 + k_x^2) - k_y k_z k_y + k_y(k_x^2 + k_y^2) \tag{84}$$

[8]Brandenburg, A., "The inverse cascade and nonlinear alpha-effect in simulations of isotropic helical hydromagnetic turbulence," *Astrophys. J.* **550**, 824-840 (2001).

Next, the terms $+k_zk_yk_z$ and $-k_yk_zk_y$ can be subsumed in the $+k_y(k_x^2+k_y^2)$ and $-k_z(k_z^2+k_x^2)$ terms, respectively, so that

$$-k^2(k_y-k_z)-k_zk^2+k_yk^2 \tag{85}$$

which cancels to zero.

Another possibility is to multiply the helicity matrix

$$H_{ij}=(\delta_{ij}-\mathrm{i}\hat{k}_k\epsilon_{ijk})/\sqrt{2} \tag{86}$$

with a vector perpendicular to $\boldsymbol{k}$, i.e., $\boldsymbol{k}\times\hat{\boldsymbol{e}}$, where $\hat{\boldsymbol{e}}$ is another arbitrary unit vector not aligned with $\boldsymbol{k}$. If we take $\hat{\boldsymbol{e}}=(1,1,1)/\sqrt{3}$, then we have

$$\boldsymbol{k}\times\hat{\boldsymbol{e}}=\frac{1}{\sqrt{3}}\begin{pmatrix}k_y-k_z\\ k_z-k_x\\ k_x-k_y\end{pmatrix} \tag{87}$$

so

$$k\begin{pmatrix}B_{0x}\\ B_{0y}\\ B_{0z}\end{pmatrix}=\begin{pmatrix}k(k_y-k_z)-\mathrm{i}k_z(k_z-k_x)+\mathrm{i}k_y(k_x-k_y)\\ k(k_z-k_x)-\mathrm{i}k_x(k_x-k_y)+\mathrm{i}k_z(k_y-k_z)\\ k(k_x-k_y)-\mathrm{i}k_y(k_y-k_z)+\mathrm{i}k_x(k_z-k_x)\end{pmatrix} \tag{88}$$

Let is check this one and insert into $kB_{0x}+\mathrm{i}k_zB_{0y}-\mathrm{i}k_yB_{0z}=0$, so

$$k^2(k_y-k_z)-\mathrm{i}k_zk(k_z-k_x)+\mathrm{i}k_yk(k_x-k_y) \tag{89}$$

$$+\mathrm{i}k_zk(k_z-k_x)+k_zk_x(k_x-k_y)-k_z^2(k_y-k_z) \tag{90}$$

$$-\mathrm{i}k_yk((k_x-k_y)-k_y^2(k_y-k_z)+k_xk_y(k_z-k_x). \tag{91}$$

Here all the imaginary terms vanish, so we are left with

$$k^2(k_y-k_z)+k_zk_x(k_x-k_y)-k_z^2(k_y-k_z)-k_y^2(k_y-k_z)+k_xk_y(k_z-k_x). \tag{92}$$

Here the fully mixed terms cancel, so

$$k^2(k_y-k_z)+k_zk_xk_x-k_z^2(k_y-k_z)-k_y^2(k_y-k_z)-k_xk_yk_x. \tag{93}$$

Here, $+k_zk_xk_x-k_xk_yk_x$ combines into $-k_x^2(k_y-k_z)$, so we have

$$k^2(k_y-k_z)-(k_z^2+k_x^2+k_y^2)(k_y-k_z)=0 \tag{94}$$