ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Problem Set 5 (Due Fri. [*not Wed.*], Nov 18, 2016) November 4, 2016, Axel Brandenburg

1. **The Sine–Gordon equation** is given by

$$\phi_{,tt} - \phi_{,xx} + \sin\phi = 0. \tag{1}$$

Verify (numerically or analytically) that

$$\phi(x,t) = 4\arctan e^{a(x-ct)} \tag{2}$$

with $\gamma = (1-c^2)^{-1/2}$ is a solution.

....

We calculate analytically $\phi_{,tt}$ and $\phi_{,xx}$ and insert. We use $\mathrm{d}(\arctan x)/\mathrm{d}x = 1/(1+x^2)$.

(a) We begin with the time derivatives:

$$\phi_{,t} = -\frac{4ace^{a(x-ct)}}{1+e^{2a(x-ct)}}. \tag{3}$$

Use product or quotient rule, so

$$\phi_{,tt} = \frac{4a^2c^2e^{a(x-ct)}}{1+e^{2a(x-ct)}} - \frac{8a^2c^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2}. \tag{4}$$

Put on the same denominator, so

$$\phi_{,tt} = \frac{4a^2c^2e^{a(x-ct)}[1+e^{2a(x-ct)}] - 8a^2c^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2}, \tag{5}$$

or

$$\phi_{,tt} = \frac{4a^2c^2e^{a(x-ct)} - 4a^2c^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2} = 4a^2c^2e^{a(x-ct)}\frac{1-e^{2a(x-ct)}}{[1+e^{2a(x-ct)}]^2}. \tag{6}$$

Computing $\phi_{,xx}$ works analogously, except that the $-ac$ prefactor from the inner derivative is replaced by just a, and a^2c^2 is replaced by a^2, so

$$\phi_{,x} = \frac{4ae^{a(x-ct)}}{1+e^{2a(x-ct)}}, \tag{7}$$

$$\phi_{,xx} = \frac{4a^2e^{a(x-ct)}}{1+e^{2a(x-ct)}} - \frac{8a^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2}. \tag{8}$$

Put on the same denominator, so

$$\phi_{,xx} = \frac{4a^2e^{a(x-ct)}[1+e^{2a(x-ct)}] - 8a^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2}, \tag{9}$$

or

$$\phi_{,xx} = \frac{4a^2e^{a(x-ct)} - 4a^2e^{3a(x-ct)}}{[1+e^{2a(x-ct)}]^2} = 4a^2e^{a(x-ct)}\frac{1-e^{2a(x-ct)}}{[1+e^{2a(x-ct)}]^2}. \tag{10}$$

Thus, we have

$$\phi_{,tt} - \phi_{,xx} = 4a^2(1-c^2)e^{a(x-ct)}\frac{1-e^{2a(x-ct)}}{[1+e^{2a(x-ct)}]^2}. \tag{11}$$

(b) To deal with this $\sin\phi$ term, where ϕ is itself an arctan, let us write $A = \tan\alpha$, so $\alpha = \arctan A$. In the end, we want the $\sin\alpha$, so let us therefore express the tangens in terms of sine (and cosine) functions, so

$$A^2 \equiv \tan^2\alpha = \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\sin^2\alpha}{1-\sin^2\alpha}, \tag{12}$$

so $A^2(1-\sin^2\alpha) = \sin^2\alpha$ or $A^2 = (1+A^2)\sin^2\alpha$, and therefore

$$\sin\alpha = \frac{A}{\sqrt{1+A^2}}. \tag{13}$$

However, there is a factor 4 in Equation (2), so we cannot apply this trick directly. Instead of working with a factor of 4, let us first work with a factor of 2 by using the formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$, or

$$\sin\phi = 2\sin\frac{\phi}{2}\,\cos\frac{\phi}{2}. \tag{14}$$

Using this again, and also $\cos 2\alpha = 1 - 2\sin^2\alpha$, we have

$$\sin\phi = 4\sin\frac{\phi}{4}\,\cos\frac{\phi}{4}\left(1-2\sin^2\frac{\phi}{4}\right). \tag{15}$$

We can now use Equation (13), because $\phi/4$ corresponds to α, but then we also need a corresponding expression for $\cos\alpha$. We can write A^2 also as $A^2 = (1-\cos^2\alpha)/\cos^2\alpha$ or $A^2\cos^2\alpha = 1-\cos^2\alpha$, and therefore $\cos\alpha = 1/\sqrt{1+A^2}$. Thus, Equation (15) becomes

$$\sin\phi = 4\sin\alpha\,\cos\alpha\left(1-2\sin^2\alpha\right) = 4\frac{A}{\sqrt{1+A^2}}\frac{1}{\sqrt{1+A^2}}\left(1-2\frac{A^2}{1+A^2}\right), \tag{16}$$

or

$$\sin\phi = \frac{4A}{1+A^2}\left(1-\frac{2A^2}{1+A^2}\right) = 4A\,\frac{1+A^2-2A^2}{(1+A^2)^2} = 4A\,\frac{1-A^2}{(1+A^2)^2}. \tag{17}$$

Equation (17) looks similar to Equation (11), so we put

$$A = e^{a(x-ct)} \tag{18}$$

and find

$$\phi_{,tt} - \phi_{,xx} = -a^2(1-c^2)\sin\phi. \tag{19}$$

Thus, Equation (1) is obeyed if $a^2(1-c^2)=1$. In other words, the propagation speed c depends (not surprisingly) on a. Writing $1-c^2 = 1/a^2$, or $c^2 = 1-1/a^2 = (a^2-1)/a^2$. To have real values of c, we better have $a>1$, and then

$$c = (1-1/a^2)^{1/2}. \tag{20}$$

In Figure 1 we show c versus a, so it is clear that larger amplitudes lead to faster propagation. This is similar to what is found for other soliton solutions.

2. **Finite difference formula.** Finite derivative formulae for the nth derivative can generally be written as

$$\frac{\mathrm{d}^n f_i}{\mathrm{d}x^n} = \frac{1}{\delta x^n}\sum_{j=-N}^{N} c_j^{(n)} f_{i+j}, \tag{21}$$

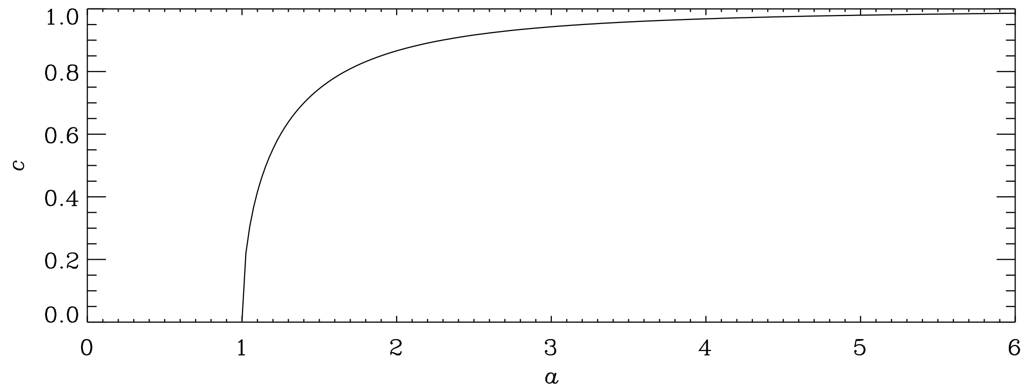

Figure 1: Plot of c versus a, so larger amplitudes lead to faster propagation.

with coefficient $c_j^{(n)}$ given in Table 1 for schemes of order N.

To derive the coefficients $c_j^{(n)}$, consider the Taylor expansion for $f_{i+j} \equiv f_i(j\delta x)$,

$$f_{i+j} = f_i + j\delta x f_i' + \tfrac{1}{2} j^2 \delta x^2 f_i'' + \tfrac{1}{3!} j^3 \delta x^3 f_i''' + \tfrac{1}{4!} j^4 \delta x^4 f_i^{\mathrm{iv}} + \tfrac{1}{5!} j^5 \delta x^5 f_i^{\mathrm{v}} + ... \tag{22}$$

(a) Write the Taylor expansion for f_{i+j} as matrix equation

$$f_{i+j} = M_{jk} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f_i, \tag{23}$$

where $M_{jk} = (j\delta x)^j$ is a matrix whose rank depends on the order of the scheme. Invert the matrix to compute the coefficients in front of the f_{i+j} for the finite difference derivative

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f_i = (M_{jk})^{-1} f_{i+j} \tag{24}$$

and verify the values given in Table 1.

Table 1: Coefficients $c_j^{(n)} \equiv a_j^{(n)}/b^{(n)}$

N	n	$b^{(n)}$	$a_0^{(n)}$	$a_1^{(n)}$	$a_2^{(n)}$	$a_3^{(n)}$	$a_4^{(n)}$	$a_5^{(n)}$
10	1	2520	0	2100	−600	150	−25	2
8	1	840	0	672	−168	32	−3	
6	1	60	0	45	−9	1		
4	1	12	0	8	−1			
2	1	2	0	1				
10	2	25200	−73766	42000	−6000	1000	−125	8
8	2	5040	−14350	8064	−1008	128	−9	
6	2	180	−490	270	−27	2		
4	2	12	−30	16	−1			
2	2	1	−2	1				

(b) Extend the table to compute the coefficients for the third derivative with a stencil width 5.

(c) What is the error of these schemes, i.e., what is the power of δx with which it scales, what is the leading derivative, and what are the coefficients. *One or two examples of your choice will be enough.*

. . . .

3. I computed the numbers in IDL, where I compute the numbers via

```
for i=0,N do begin
for j=0,N do begin
  ii=i-.5*N
  mat(i,j)=ii^j/factorial(j)
endfor
endfor
```

and then I inverted the matrix with `m1=invert(mat)` and printed the result. I tried a few integer denominators to get the fraction.

4. Using the same method, the numbers for $N=3$ have been computed; see the results below.

Table 2: Coefficients $c_j^{(n)} \equiv a_j^{(n)}/b^{(n)}$ for $n=3$

N	n	$b^{(n)}$	$a_0^{(n)}$	$a_1^{(n)}$	$a_2^{(n)}$	$a_3^{(n)}$	$a_4^{(n)}$	$a_5^{(n)}$
10	3	30240	0	-70098	52428	-14607	2522	-205
8	3	240	0	-488	338	-72	7	
6	3	8	0	-13	8	-1		
4	3	2	0	-2	1			

5. When assembling the derivative with neighboring points using Equation (23), the low order derivative terms in the Taylor expansion all cancel. The lowest one that does not cancel is proportional to the δx to the power n, where n is the order of the scheme. It is then also proportional to a higher derivative, whose order

is clear from dimensional arguments, for example for a 2nd-order scheme, to leading order, the error in f' is proportional to the 3rd derivative, $(\delta x)^2\, f'''$, and for f'' it is proportional to the 4th derivative, $(\delta x)^4\, f^{(iv)}$. Likewise, for a 10th-order scheme, to leading order, the error in f' is proportional to the 11th derivative, $(\delta x)^{10}\, f^{(xi)}$, and for f'' it is proportional to the 12th derivative, $(\delta x)^{10}\, f^{(xii)}$. To compute the coefficient in front of it, we simply add the corresponding contributions (with the factors given in Table 1, that enter in Equation (24)). We find:

$$(f')_{\rm 2nd} = f' + 2\,\frac{1}{2\delta x}\,\frac{(\delta x)^3}{3!}\,f''' = f' + 2\,\frac{(\delta x)^2}{2\times 3!}\,f''' = f' + \frac{(\delta x)^2}{6}\,f'''. \tag{25}$$

The factor of two accounts for the contributions from the left and the right. Next,

$$(f')_{\rm 4th} = f' + 2\,\frac{8-2^5}{12\delta x}\,\frac{(\delta x)^5}{5!}\,f^{(\rm v)} = f' + 2\,\frac{8-2^5}{12\times 5!}\,(\delta x)^4\,f^{(\rm v)} = f' - \frac{1}{30}\,(\delta x)^4\,f^{(\rm v)}. \tag{26}$$

Now that we get the hang of it, we'll skip the intermediate step, so

$$(f')_{6\text{th}} = f' + 2\,\frac{45 - 9\times 2^7 + 3^7}{60\times 7!}\,(\delta x)^6\, f^{(\text{vii})} = f' + \frac{1}{140}\,(\delta x)^6\, f^{(\text{vii})}, \tag{27}$$

$$(f')_{8\text{th}} = f' + 2\,\frac{672 - 168\times 2^9 + 32\times 3^9 - 3\times 4^9}{840\times 9!}\,(\delta x)^8\, f^{(\text{ix})} = f' - \frac{1}{630}\,(\delta x)^8\, f^{(\text{ix})}, \tag{28}$$

$$(f')_{10\text{th}} = f' + 2\,\frac{2100 - 600\times 2^{11} + 150\times 3^{11} - 25\times 4^{11} + 2\times 5^{11}}{2520\times 11!}\,(\delta x)^{10}\, f^{(\text{xi})}, \tag{29}$$

where the coefficient is $+1/2772$. Note that in all these cases, the leading error is dispersive. For the second derivative, the leading error is diffusive, but this is unimportant because the diffusion caused by the 2nd derivative itself is more important. So, for f'' we find

$$(f'')_{2\text{nd}} = f'' + 2\,\frac{1}{(\delta x)^2}\,\frac{(\delta x)^4}{4!} f^{(\text{iv})} = f'' + 2\,\frac{(\delta x)^2}{4!} f^{(\text{iv})} = f'' + \frac{(\delta x)^2}{12} f^{(\text{iv})}, \tag{30}$$

$$(f'')_{4\text{th}} = f'' + 2\,\frac{16 - 2^6}{12\times 6!}\,(\delta x)^4\, f^{(\text{vi})} = f'' - \frac{(\delta x)^4}{90} f^{(\text{vi})}, \tag{31}$$

$$(f'')_{6\text{th}} = f'' + 2\,\frac{270 - 27\times 2^8 + 2\times 3^8}{180\times 8!}\,(\delta x)^6\, f^{(\text{viii})} = f'' + \frac{(\delta x)^6}{560} f^{(\text{viii})}, \tag{32}$$

$$(f'')_{8\text{th}} = f'' + 2\,\frac{8064 - 1008\times 2^{10} + 128\times 3^{10} - 9\times 4^{10}}{5040\times 10!}\,(\delta x)^8\, f^{(\text{x})} = f'' - \frac{(\delta x)^8}{3150} f^{(\text{x})}, \tag{33}$$

$$(f'')_{10\text{th}} = f'' + 2\,\frac{42000 - 6000\times 2^{12} + 1000\times 3^{12} - 125\times 4^{12} + 8\times 5^{12}}{25200\times 12!}\,(\delta x)^{10}\, f^{(\text{xii})}, \tag{34}$$

where the coefficient is $+1/16632$. This same scheme can be continued for the 3rd derivatives,

$$(f''')_{2\text{nd}} = f''' + 2\,\frac{-2 + 2^5}{2\,(\delta x)^3}\,\frac{(\delta x)^5}{5!} f^{(\text{v})} = f''' + 2\,\frac{-2 + 2^5}{2\times 5!}\,(\delta x)^2\, f^{(\text{v})} = f''' + \frac{1}{4}\,(\delta x)^2\, f^{(\text{v})}. \tag{35}$$

$$(f''')_{4\text{th}} = f''' + 2\,\frac{-13 + 8\times 2^7 - 3^6}{8\times 7!}\,(\delta x)^4\, f^{(\text{vii})} = f''' - \frac{294}{7!}\,(\delta x)^4\, f^{(\text{vii})}, \tag{36}$$

where the prefactor reduces to $-7/(4\times 5\times 6) \approx -0.058$, and so forth. The leading error is thus proportional to the $n+N$th derivative.

6. **Second-next nearest neighbor shell model.** In Handout 15b, a shell model with nearest neighbors was presented. A shell model with second-next nearest neighbors allows us to conserve *two* conservation laws. The models are sometimes called GOY models to acknowledge the work of Gledzer, Ohkitani, and Yamada.[1]

   (a) Show that the general form of such a model is

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \mathrm{i}k_n\left(Au_{n-2}u_{n-1} + Bu_{n-1}u_{n+1} + Cu_{n+1}u_{n+2}\right)^* - \nu k_n^2 u_n. \tag{37}$$

   where $k_n = k_0\, 2^n$ is the wavenumber shell. The asterisk means complex conjugation.

[1] E. B. Gledzer, "System of hydrodynamic type admitting two quadratic integrals of motion," *Sov. Phys. Dokl.* **18**, 216 (1973). M. Yamada & K. Ohkitani, "Lyapunov spectrum of a model of two-dimensional turbulence," *Phys. Rev. Lett.* **60**, 983-986 (1988).

(b) Assume that both energy $E = \frac{1}{2}\sum |u_n|^2$ and enstrophy $\Xi = \frac{1}{2}\sum k_n^2 |u_n|^2$ are conserved, and show that

$$A = -\tfrac{1}{10}. \quad B = 1, \quad C = -\tfrac{8}{5}. \tag{38}$$

(c) Next, assume that both energy $E = \frac{1}{2}\sum u_n^2$ and helicity $H = \frac{1}{2}\sum (-1)^n k_n |u_n|^2$ are conserved, and show that

$$A = \tfrac{1}{2}, \quad B = 1, \quad C = -4. \tag{39}$$

. . . .

(a) To obey any conservation law with second-nearest neighbors, we must consider all triples that contribute in $\sum u_n N_n(u,u)$. Here,

$$N_n(u,u) = \sum_{i=-2}^{2}\sum_{i=-2}^{2} u_{n+i}u_{n+j}. \tag{40}$$

Let us begin with a list of triples (n,i,j) that can contribute.

```
n=-2
                      (-2,-2,-2) (-2,-2,-1) (-2,-2, 0)
                      (-2,-1,-2) (-2,-1,-1) (-2,-1, 0)
                      (-2, 0,-2) (-2, 0,-1) (-2, 0, 0)

n=-1
           (-1,-2,-2) (-1,-2,-1) (-1,-2, 0) (-1,-2, 1)
           (-1,-1,-2) (-1,-1,-1) (-1,-1, 0) (-1,-1, 1)
           (-1, 0,-2) (-1, 0,-1) (-1, 0, 0) (-1, 0, 1)
           (-1, 1,-2) (-1, 1,-1) (-1, 1, 0) (-1, 1, 1)

n=0
( 0,-2,-2) ( 0,-2,-1) ( 0,-2, 0) ( 0,-2, 1) ( 0,-2, 2)
( 0,-1,-2) ( 0,-1,-1) ( 0,-1, 0) ( 0,-1, 1) ( 0,-1, 2)
( 0, 0,-2) ( 0, 0,-1) ( 0, 0, 0) ( 0, 0, 1) ( 0, 0, 2)
( 0, 1,-2) ( 0, 1,-1) ( 0, 1, 0) ( 0, 1, 1) ( 0, 1, 2)
( 0, 2,-2) ( 0, 2,-1) ( 0, 2, 0) ( 0, 2, 1) ( 0, 2, 2)

n=1
( 1,-1,-1) ( 1,-1, 0) ( 1,-1, 1) ( 1,-1, 2)
( 1, 0,-1) ( 1, 0, 0) ( 1, 0, 1) ( 1, 0, 2)
( 1, 1,-1) ( 1, 1, 0) ( 1, 1, 1) ( 1, 1, 2)
( 1, 2,-1) ( 1, 2, 0) ( 1, 2, 1) ( 1, 2, 2)

n=2
( 2, 0, 0) ( 2, 0, 1) ( 2, 0, 2)
( 2, 1, 0) ( 2, 1, 1) ( 2, 1, 2)
( 2, 2, 0) ( 2, 2, 1) ( 2, 2, 2)
```

Out of these many triples, there are all those that we used in the nearest neighbor model. However, they all have a problem in that they violate the Liouville theorem, which states that the phase space volume must be constant. This in turn means that $\partial N_n(u,u)/\partial u_n = 0$, i.e., the $N_n(u,u)$ should only contain terms with $u_{n\pm1}$ and $u_{n\pm2}$, but no u_n terms.

Thus, having masked out the terms that are incompatible with the Liouville theorem, we are left with the following set of triples:

```
n=-2
                                 .  .  .    .  .  .
                                 .  .  .    .  .  .
                                 .  .  .    .  .  .
                               (-2,-1,-1) (-2,-1, 0)*
                               (-2, 0,-1) (-2, 0, 0)

n=-1
             .  .  .               .  .  .    .  .  .
           (-1,-2,-2)          (-1,-2, 0) (-1,-2, 1)
             .  .  .               .  .  .    .  .  .
           (-1, 0,-2)*         (-1, 0, 0) (-1, 0, 1)
           (-1, 1,-2)          (-1, 1, 0) (-1, 1, 1)

n=0
( 0,-2,-2) ( 0,-2,-1)*         ( 0,-2, 1) ( 0,-2, 2)
( 0,-1,-2) ( 0,-1,-1)          ( 0,-1, 1) ( 0,-1, 2)
  .  .  .    .  .  .             .  .  .    .  .  .
( 0, 1,-2) ( 0, 1,-1)          ( 0, 1, 1) ( 0, 1, 2)
( 0, 2,-2) ( 0, 2,-1)          ( 0, 2, 1) ( 0, 2, 2)

n=1
( 1,-1,-1) ( 1,-1, 0)          ( 1,-1, 2)
( 1, 0,-1) ( 1, 0, 0)          ( 1, 0, 2)
  .  .  .    .  .  .             .  .  .
( 1, 2,-1) ( 1, 2, 0)          ( 1, 2, 2)
  .  .  .    .  .  .

n=2
( 2, 0, 0) ( 2, 0, 1)
( 2, 1, 0) ( 2, 1, 1)
  .  .  .    .  .  .
  .  .  .    .  .  .
```

There are three terms with an astersk that we shall discuss in a moment, but let us start at the beginning, where the first term is $(-2,-1,-1)$. The position where this it occurs corresponds to the term u_{n+1}^2, because $n=-2$. However, the possible interaction partners $(-1,-2,-1)$ and $(-1,-1,-2)$ corresponds both to $u_n u_{n+1}$, which is not allowed by Liouville's theorem.

The next term is $(-2,-1,0)$, which corresponds to $u_{n+1}u_{n+2}$. It has the following two partners: $(0,-2,-1)$ and $(-1,0,-2)$, which correspond to $u_{n-1}u_{n-2}$ and $u_{n-1}u_{n+1}$, respectively. Those are the standard terms in the GOY model, and are marked by the asterisks above.

It turns out that all other triples in which no two members are the same are equivalent to the GOY terms, and those where two members are the same all have problems with

Liouville's theorem.

(b) Energy conservation means that $u_n \mathrm{d}u_n^*/\mathrm{d}t = 0$. Thus, with the three terms isolated above, we have

$$\left[u_{n'}k_{n'}Cu_{n'+1}u_{n'+2}\right|_{n'=n-2}\left[u_{n'}k_{n'}Bu_{n'+1}u_{n'-1}\right|_{n'=n-1}\left[u_{n'}k_{n'}Au_{n'-2}u_{n'-1}\right|_{n'=n} = 0. \quad (41)$$

Thus, we have

$$u_{n-2}k_{n-2}Cu_{n-1}u_n + u_{n-1}k_{n-1}Bu_nu_{n-2} + u_nk_nAu_{n-2}u_{n-1} = 0, \quad (42)$$

or

$$k_{n-2}C + k_{n-1}B + k_nA = 0. \quad (43)$$

Enstrophy conservation implies

$$k_{n-2}^3C + k_{n-1}^3B + k_n^3A = 0. \quad (44)$$

Using $k_{n-1} = k_n/2$, we have

$$C/4 + B/2 + A = 0, \quad (45)$$

$$C/4^3 + B/2^3 + A = 0. \quad (46)$$

Since the right-hand side can be multiplied by an arbitrary factor, we can assume $B = 1$. Subtracting the two equations yields

$$\frac{16-1}{64}C + \frac{4-1}{8} = 0. \quad (47)$$

Multipling by 8 yields $\frac{15}{8}C = -3$, so $\frac{5}{8}C = -1$, or $C = -\frac{8}{5}$. Next, we have $A = -C/4 - 1/2 = \frac{2}{5} - 1/2 = -1/10$.

(c) If we have helicity conservation, we have instead of Equation (46)

$$C/4^2 - B/2^2 + A = 0 \quad (48)$$

together with Equation (46). Again, subtracting the two equations yields

$$\frac{4-1}{16}C + \frac{2+1}{4} = 0. \quad (49)$$

Multipling by 4 yields $\frac{3}{4}C = -3$, so $\frac{1}{4}C = -1$, or $C = -4$. Next, we have $A = -C/4 - 1/2 = 1 - 1/2 = 1/2$.