ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Midterm exam, Oct 12, 2016 October 17, 2016, Axel Brandenburg

1. **Magneto-rotational instability.**

   (a) Explain qualitatively, in a few sentences, the magneto-rotational instability. Remember the properties of Keplerian orbits. Use a sketch!

   (b) What is the source of energy?

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   (a) The restoring force is the tension force, i.e., the contribution $\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}$ as part of the Lorentz force $(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}$. It normally produces slow magnetosonic waves, but when the Alfvén frequency, $\omega_\mathrm{A}=v_\mathrm{A}k$, becomes comparable with and weaker than the orbital frequency, Ω, its character changes and these waves become unstable. This is because two points along a field line try a attract each other, so the part that lags behind has to speed up. But the required acceleration brings this fluid parcel to a higher orbit, i.e., with a larger radius, This orbit is slower, so this part continues to lag behind even more, the separation increases further and there is a run-away.

   (b) The source of energy is ultimately potential energy, $-GM/r$, where M is the mass of the central object.[1] At the level of the shearing box approximation, the relevant energy form is the large-scale shear flow. Magnetic energy plays no direct role, because the magnetic energy is dissipated into heat. In that sense, magnetic fields act merely like a catalyst.

2. **Shearing box approximation.**

   (a) Explain the limitations of the shearing box approximation. Use a sketch indicating the position of the central object.

   (b) Give the terms that show at what rate kinetic and magnetic energy is being tapped. For this, recall the governing equations in the form

$$\frac{\partial\boldsymbol{u}}{\partial t}+Sx\frac{\partial\boldsymbol{u}}{\partial y}+u_xS\hat{\boldsymbol{y}}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}+2\boldsymbol{\Omega}\times\boldsymbol{u}=-\rho^{-1}\boldsymbol{\nabla}P+\rho^{-1}\boldsymbol{J}\times\boldsymbol{B}, \tag{1}$$

$$\frac{\partial\boldsymbol{B}}{\partial t}+Sx\frac{\partial\boldsymbol{B}}{\partial y}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B}=B_xS\hat{\boldsymbol{y}}+\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{2}$$

$$\frac{\partial\rho}{\partial t}+Sx\frac{\partial\rho}{\partial y}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho=-\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{3}$$

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   (a) One cannot distinguish between left and right; see the sketch. This is because there are no curvature terms in this approximation.[2]

[1] The resulting luminosity is $L=GM\dot{M}/2r$, where $\dot{M}$ is the accreted mass. The factor 2 comes from the fact that the other half of the energy remains in the form of kinetic energy. Most of the energy is released at small radii, which becomes particularly efficient if the central body is small enough, i.e., a black hole. Inserting the Schwarzschild radius, $r_S=3\times2GM/c^2$, we have $L=\dot{M}c^2/12\approx0.08\dot{M}c^2$. (This is 12 times more efficient that nuclear fusion, which yields $L\approx0.007\dot{M}c^2$.)

[2] Axisymmetry is not a limitation. In fact, three-dimensional solutions are necessary if one wants to simulate a self-excited system where the magnetic field is being constantly regenerated by dynamo action. In the linear analysis, we neglected y derivatives just for technical reasons, because then we can't make the $e^{\mathrm{i}k_yy}$ ansatz. What one could do is $e^{\mathrm{i}k_y(t)y}$.

(b) The qualitatively new terms are what one might call "stretching terms",

$$\frac{\partial \boldsymbol{u}}{\partial t} = -u_x S \hat{\boldsymbol{y}} + ...$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = B_x S \hat{\boldsymbol{y}} + ...$$

To compute the kinetic energy equation, we multiply by $\rho \boldsymbol{u}$, so we find

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho \boldsymbol{u}^2\right) = -\rho u_x u_y S + ...$$

and likewise for the magnetic energy equation we have

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\boldsymbol{B}^2/\mu_0\right) = -(B_x B_y/\mu_0) S + ...$$

The terms on the right-hand side are the Reynolds and Maxwell stresses. Thus, to find the energy conversion rate, one needs to compute these stresses.

3. **Inflection point instability.**

(a) Use the sketch in Fig. 1 to explain the inflection point instability.

(b) Explain qualitatively what happens in the nonlinear regime in the presence of viscosity. Again, use s sketch.

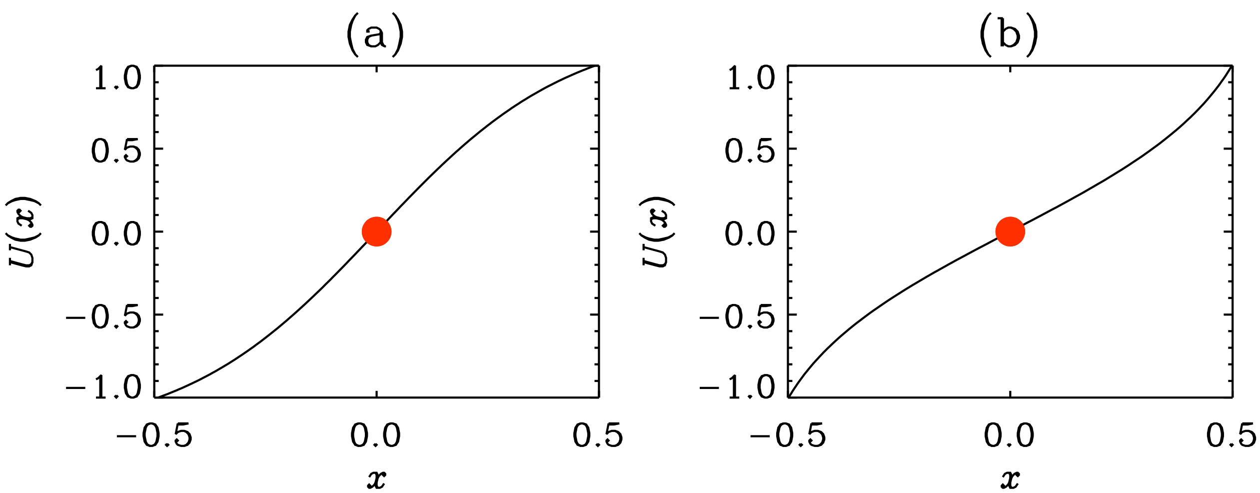

Figure 1: Sketch of two possible shear flow profiles. Both have an inflection point marked in red.

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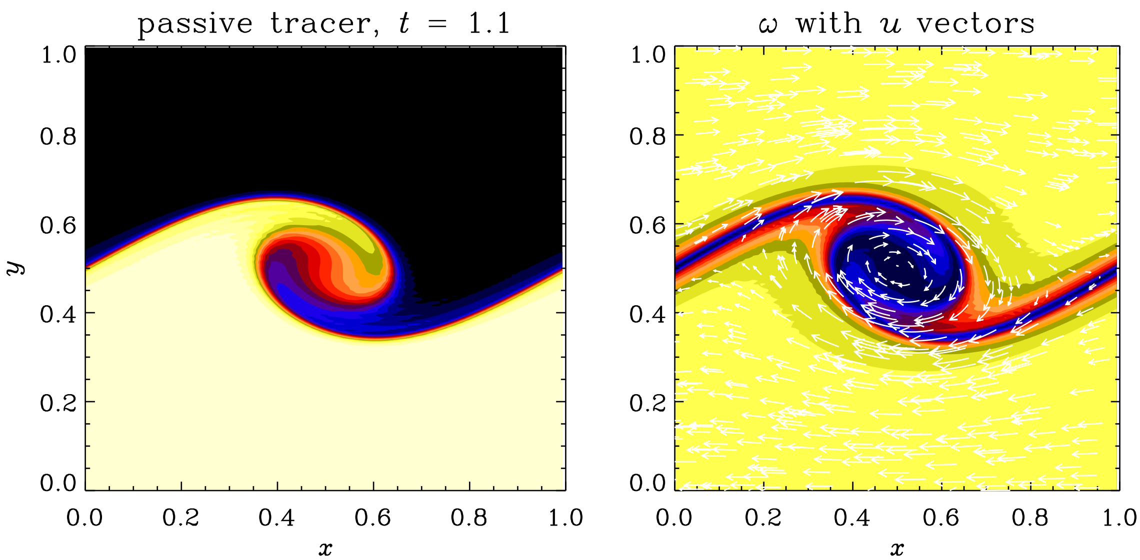
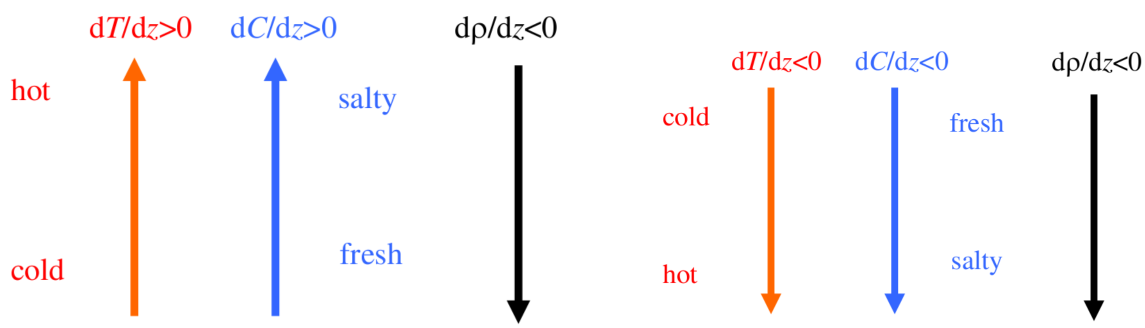
(a) By Rayleigh's inflection point theorem, a necessary condition for instability is the presence of an inflection point within the domain, i.e., $U'' = 0$. This is the case for both plots. In addition, by Fjørtoft's theorem, we require $(U - U_S)U'' < 0$, where U_S is the speed at the location of the inflection point (which is zero in this sketch), The difference between the two sketches is the sense of curvature, i.e., the sign of U''. For $x > 0$, where $U - U_S > 0$ it is negative in sketch (a) and positive in (b). Thus, only case (a) obeys the necessary condition for Fjørtoft's theorem. The situation is analogous for $x < 0$, where $U - U_S < 0$, but now U'' is positive in sketch (a) and negative in sketch (b).

(b) Already in the linear regime, small patches of positive and negative vorticity perturbation occur. These patches intensify in the nonlinear regime, which leads to the roll-up of the entire vortex sheet. This was demonstrated in Handout 10 using a nonlinear fluid dynamics code.



4. **Double-diffusive instability.**

Consider a stratified fluid where the fluid density increases downward, but perturbations in the density are affected by both the temperature and the salt concentration.



(a) What are the stability properties of the fluid on the left?

(b) What are the stability properties of the fluid on the right?

(c) What is meant by overstability?

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(a) Although temperature is stably stratified, the concentration is unstably stratified, so the system can become unstable if the latter effect dominates over the former. This leads to *thermo-haline convection* and the phenomenon of "fingering", as is also shown in Fig. 11.14 of Kundu et al.

(b) Now the temperature is unstably stratified, but the concentration is stably stratified, so it tends to stabilize the system. However, it does so too much: if fluid moves downward, fresh water moves into the denser deeper parts and bounces back and overshoots the point where it came from. This leads to what is called overstability. The resulting flow is sometimes called *semi-convection* and it is important in the cores of massive stars, which are convectively unstable by Schwarzschild's criterion, but stable by the Ledoux criterion.

(c) Overstability refer to the case where the imaginary part of σ is nonvanishing ($\rightarrow$ oscillatory behavior) and the real part of σ becomes positive ($\rightarrow$ unstable, exponential growth).

5. **Acoustic cavity.** Consider the dispersion relation for sound waves in an isothermal stratified layer,

$$\omega^2 = \tfrac{1}{2}\frac{g^2}{c_{\rm s}^2} + c_{\rm s}^2 \boldsymbol{k}^2 \tag{4}$$

where $\boldsymbol{k}^2 = \boldsymbol{k}_\perp^2 + k_\parallel^2$ with $\boldsymbol{k}$ being the wavevector, $\boldsymbol{k}_\perp$ the horizontal wavevector and $k_\parallel$ the wavenumber in the vertical direction.

(a) Rewrite this as an equation for $k_\parallel$. Assume that we can interpret this equation, at least approximately, in the case of nonuniform sound speed $c_{\rm s} = c_{\rm s}(z)$, where z is the vertical coordinate. Assume that $c_{\rm s}$ increases with depth. Explain in which depth range do you expect waves to be non-evanescent? Give expressions for $c_{\rm s}$ for the critical values where waves are non-evanescent.

(b) Assume $\omega = 0.02\,{\rm s}^{-1}$ and $k_\perp = 100/R_\odot$, where $R_\odot = 700\,{\rm Mm}$ is the solar radius. Compute one of the critical points. Why does the value of g not matter here? Where approximately in the Sun this point? For orientation, recall that at the solar surface $c_{\rm s} = 6\,{\rm km\,s^{-1}}$ and at the bottom of the solar convection zone we have $c_{\rm s} = 200\,{\rm km\,s^{-1}}$.

(c) Assume $\omega = 0.02\,{\rm s}^{-1}$ and $g = 300\,{\rm m\,s^{-1}}$, and compute the critical value of $c_{\rm s}$. Compute another one of the critical points. Why does the value of $k_\perp$ not matter here? Where approximately in the Sun this point? Again, for orientation, at the solar surface we have $c_{\rm s} = 6\,{\rm km\,s^{-1}}$, and $c_{\rm s} = 200\,{\rm km\,s^{-1}}$ at the bottom of the solar convection zone.

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(a) Moving the first term on the right to the other side, and inserting $\boldsymbol{k}^2 = \boldsymbol{k}_\perp^2 + k_\parallel^2$ gives

$$\frac{\omega^2 - g^2/2c_{\rm s}^2}{c_{\rm s}^2} = \boldsymbol{k}_\perp^2 + k_\parallel^2$$

Solving for $k_\parallel$ yields

$$k_\parallel = \sqrt{\frac{\omega^2 - g^2/2c_{\rm s}(z)^2}{c_{\rm s}(z)^2} - \boldsymbol{k}_\perp^2}$$

We expect non-evanscent waves when $k_\parallel$ is real, i.e., the term underneath the square root is positive. First of all, $\omega^2 - g^2/2c_{\rm s}(z)^2 > 0$ requires $c_{\rm s}(z)^2 > g^2/2\omega^2$, or $c_{\rm s}(z) \gtrsim 0.7g/\omega$. Furthermore, when $c_{\rm s}(z)$ becomes large, the $\boldsymbol{k}_\perp$ becomes important, which then requires $c_{\rm s}(z)^2 < \omega^2/\boldsymbol{k}_\perp^2$, i.e., $c_{\rm s}(z) < \omega/k_\perp$. Thus, we have

$$0.7g/\omega \lesssim c_{\rm s}(z) < \omega/k_\perp.$$

This yields a cavity for acoustic waves, which implies discrete frequencies.

(b) The critical sound speed is given by $c_\mathrm{s} = \omega/k_\perp = 0.02\,\mathrm{s}^{-1} \times 700\,\mathrm{Mm}/100 = 0.14\,\mathrm{Mm\,s}^{-1} = 140\,\mathrm{km\,s}^{-1}$. This is still just above the bottom of the convection zone. The g term is then unimportant, because $g/c_\mathrm{s} = (300/140{,}000)\,\mathrm{s}^{-1} = 0.002\,\mathrm{s}^{-1}$ is much (10 times) less than ω.

(c) The critical sound speed is now given by $c_\mathrm{s} = 0.7g/\omega = 0.7 \times 300/0.02\,\mathrm{m\,s}^{-1} = 10.5\,\mathrm{km\,s}^{-1}$. The $k_\perp$ does not matter now because $c_\mathrm{s}k_\perp = 10\,\mathrm{km\,s}^{-1} \times 100/(700\,\mathrm{Mm}) = 10/7000\,\mathrm{s}^{-1} = 0.002\,\mathrm{s}^{-1}$ is much less than ω,

6. **Index notation.** Using index notation, rewrite $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})$ using product rule and the fact that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{5}$$

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Write

$$[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}u_lB_m) = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})(u_lB_m)_{,j} = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})(u_{l,j}B_m+u_lB_{m,j})$$

Thus, we have

$$[\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})]_i = (u_{i,j}B_j + u_iB_{j,j}) - (u_{j,j}B_i + u_jB_{i,j})$$

Now, $u_{i,j}B_j = B_ju_{i,j} = [(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{u}]_i$, $B_{j,j} = \boldsymbol{\nabla}\cdot\boldsymbol{B}$, $u_{j,j} = \boldsymbol{\nabla}\cdot\boldsymbol{u}$, and $B_{i,j}u_j = u_jB_{i,j} = [(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{B}]_i$, so

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{u}.$$

Using Maxwell's equations and incompressibility, this can be simplified further.[3]

7. **Instability in the presence of a cosmic ray current.** In the presence of a cosmic ray current, $\boldsymbol{J}_0 = J_0\hat{\boldsymbol{z}} =$ const, the simplified (pressureless) linearized MHD equations can be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B}_0 - \boldsymbol{J}_0 \times \boldsymbol{b} \tag{6}$$

$$\frac{\partial \boldsymbol{a}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B}_0 \tag{7}$$

where $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}} =$ const is an imposed magnetic field. Lower case symbols denote small departures from the equilibrium state: $\boldsymbol{u}$ is the velocity perturbation, $\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{b}$ is the current density perturbation, $\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{a}$ is the magnetic field perturbation, and $\boldsymbol{a}$ is the magnetic vector potential perturbation.

[3] Using Maxwell's equations, we have $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Furthermore, in the incompressible case, we have $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, so $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B}$. Therefore, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} + ...$$

or

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + ...$$

which can also be written as in terms of the advective derivative

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + ...$$

The dots denote resistive terms.

(a) Assume all perturbed quantities to be proportional to $e^{\mathrm{i}kz+\sigma t}$ and verify that

$$\boldsymbol{u}\times\boldsymbol{B}_0 = \begin{pmatrix} u_y \\ -u_x \\ 0 \end{pmatrix} B_0, \quad \boldsymbol{j}\times\boldsymbol{B}_0 = \begin{pmatrix} j_y \\ -j_x \\ 0 \end{pmatrix} B_0, \quad \boldsymbol{J}_0\times\boldsymbol{b} = \begin{pmatrix} -b_y \\ b_x \\ 0 \end{pmatrix} J_0, \quad \boldsymbol{b} = \mathrm{i}k \begin{pmatrix} -a_y \\ a_x \\ 0 \end{pmatrix}, \tag{8}$$

(b) Next, show that $\boldsymbol{j} = k^2\boldsymbol{a}$, together with

$$\boldsymbol{j}\times\boldsymbol{B}_0 = \begin{pmatrix} a_y \\ -a_x \\ 0 \end{pmatrix} k^2 B_0, \quad \boldsymbol{J}_0\times\boldsymbol{b} = \mathrm{i}kJ_0 \begin{pmatrix} -a_x \\ -a_y \\ 0 \end{pmatrix}. \tag{9}$$

(c) Define the state vector $\boldsymbol{q} = (u_x, u_y, a_x, a_y)^T$ (column vector) for the matrix eigenvalue problem $\mathsf{M}\boldsymbol{q} = 0$ and show that the relevant matrix is given by

$$\mathsf{M} = \begin{pmatrix} \sigma & 0 & -\mathrm{i}kJ_0 & -k^2B_0 \\ 0 & \sigma & k^2B_0 & -\mathrm{i}kJ_0 \\ 0 & -B_0 & \sigma & 0 \\ B_0 & 0 & 0 & \sigma \end{pmatrix} \tag{10}$$

(d) Show that $\det\mathsf{M} = 0$ yields the dispersion relation in the form

$$\sigma^4 + 2k^2B_0^2\sigma^2 - k^2J_0^2B_0^2 + k^4B_0^4 = 0 \tag{11}$$

(e) Solve the biquadratic equation for σ and sketch the solution branches. What is the criterion for instability?

(f) Verify that, in the unstable case, the eigenvector is given by

$$\boldsymbol{q} = \begin{pmatrix} -(\sigma/k)\sin kz \\ (\sigma/k)\cos kz \\ (B_0/k)\cos kz \\ (B_0/k)\sin kz \end{pmatrix} e^{\sigma t}, \tag{12}$$

Hint: you could do this by showing that $\boldsymbol{j}\times\boldsymbol{B}_0 - \boldsymbol{J}_0\times\boldsymbol{b} = (-\sin kz, \cos kz)(J_0B_0 - kB_0^2)$. and $\dot{\boldsymbol{u}} = (-\sin kz, \cos kz)\sigma^2/k$. For the uncurled induction equation, you may want to demonstrate that

$$\dot{\boldsymbol{a}} = \begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} \frac{\sigma B_0}{k}, \quad \boldsymbol{u}\times\boldsymbol{B}_0 = \begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} \frac{\sigma B_0}{k}. \tag{13}$$

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(e) We define $\omega_\mathrm{A} = v_\mathrm{A}k$ and $\sigma_\mathrm{cr} = J_0$, so we have

$$\sigma_\pm^2 = -\omega_\mathrm{A}^2 \pm \sigma_\mathrm{cr}\omega_\mathrm{A} \tag{14}$$

We are mainly interested in the upper sign, $\sigma^2 = \omega_\mathrm{A}^2(|\sigma_\mathrm{cr}/\omega_\mathrm{A}|-1)$, so $\sigma = |\omega_\mathrm{A}|\sqrt{|\sigma_\mathrm{cr}/\omega_\mathrm{A}|-1}$.

(f) We have $\boldsymbol{a} = (\cos kz, \sin kz)B_0/k$, $\boldsymbol{b} = -(\cos kz, \sin kz)B_0$, $\boldsymbol{j} = (\cos kz, \sin kz)kB_0$.

$$\boldsymbol{j}\times\boldsymbol{B}_0 = \begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} kB_0 = \begin{pmatrix} \sin kz \\ -\cos kz \\ 0 \end{pmatrix} kB_0^2 = \begin{pmatrix} -\sin kz \\ \cos kz \\ 0 \end{pmatrix} \left(-kB_0^2\right), \tag{15}$$

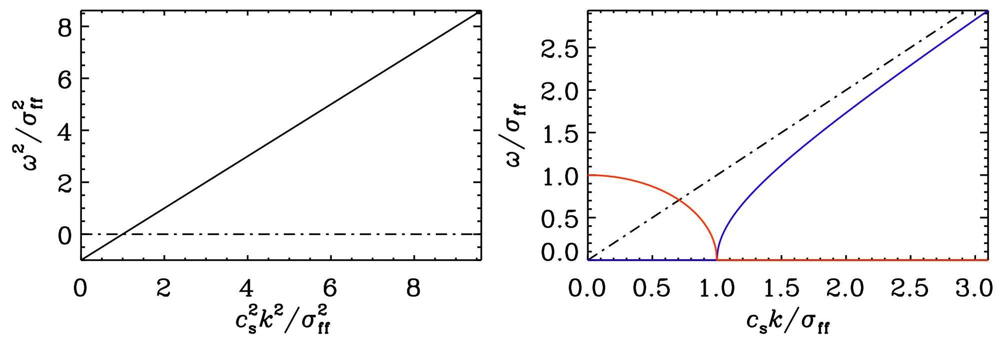

Figure 2: Dispersion relation showing $\omega^2(c_s^2k^2)/\sigma_{\rm ff}^2$ (left) and $\omega(c_sk)/\sigma_{\rm ff}$ (right), where $\sigma_{\rm ff}^2 = 4\pi G\rho_0$ has been introduced. In the right-hand panel, imaginary (real) parts are plotted in red (blue). On the left, the dispersion relation for sound waves is plotted as a dash-dotted line.

$$-\boldsymbol{J}_0 \times \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ J_0 \end{pmatrix} \times \begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} B_0 = \begin{pmatrix} -\sin kz \\ \cos kz \\ 0 \end{pmatrix} J_0 B_0, \tag{16}$$

So: $\boldsymbol{j} \times \boldsymbol{B}_0 - \boldsymbol{J}_0 \times \boldsymbol{b} = (-\sin kz, \cos kz)(J_0B_0 - kB_0^2) = (\sin kz, -\cos kz)\omega_{\rm A}(\omega_{\rm A} - \sigma_{\rm cr})$.
Next: $\boldsymbol{u} = (\cos kz, \sin kz)\omega/k$, so $\dot{\boldsymbol{u}} = (\sin kz, -\cos kz)(\omega^2/k)$,
which is equal to $\boldsymbol{j} \times \boldsymbol{B}_0 - \boldsymbol{J}_0 \times \boldsymbol{b}$ when $\omega^2 = \omega_{\rm A}^2 - \sigma_{\rm cr}\omega_{\rm A}$. Also

$$\dot{\boldsymbol{a}} = \frac{\partial}{\partial t}\begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} \frac{B_0}{k} = \begin{pmatrix} \sin kz \\ -\cos kz \\ 0 \end{pmatrix} \frac{\omega B_0}{k}, \quad \boldsymbol{u}\times\boldsymbol{B}_0 = \frac{\omega}{k}\begin{pmatrix} \cos kz \\ \sin kz \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} = \begin{pmatrix} \sin kz \\ -\cos kz \\ 0 \end{pmatrix} \frac{\omega B_0}{k}. \tag{17}$$