ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence
Preparation for midterm exam on Oct 12, 2016 October 11, 2016, Axel Brandenburg

This is a mock midterm exam; the actual exam questions will be slightly less involved and should be manageable within 50 minutes. You are welcome to email me with questions and an estimate of how long you have been struggling with each question.

1. **Thermal instability.** The energy equation can be formulated as an evolution equation for the specific entropy,

$$\rho T \frac{\mathrm{D}S}{\mathrm{D}t} = -\rho\mathcal{L} + \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T), \tag{1}$$

where $\mathcal{L} = \rho\Lambda - \Gamma$ is the difference between heating and cooling per unit mass, T is the temperature, S is the specific entropy, and K is the radiative conductivity.

(a) Give a qualitative sketch showing a possible form of $\mathcal{L}(T)$ that leads to an instability.

(b) Mark fixed points in $\mathcal{L}(T)$ and explain in words which one(s) is/are stable and which one(s) is/are unstable.

(c) Explain in words why the consideration of $\mathcal{L}(T)$ alone gives an incomplete picture and why the consideration of density and pressure changes might imply a more stringent criterion of instability.

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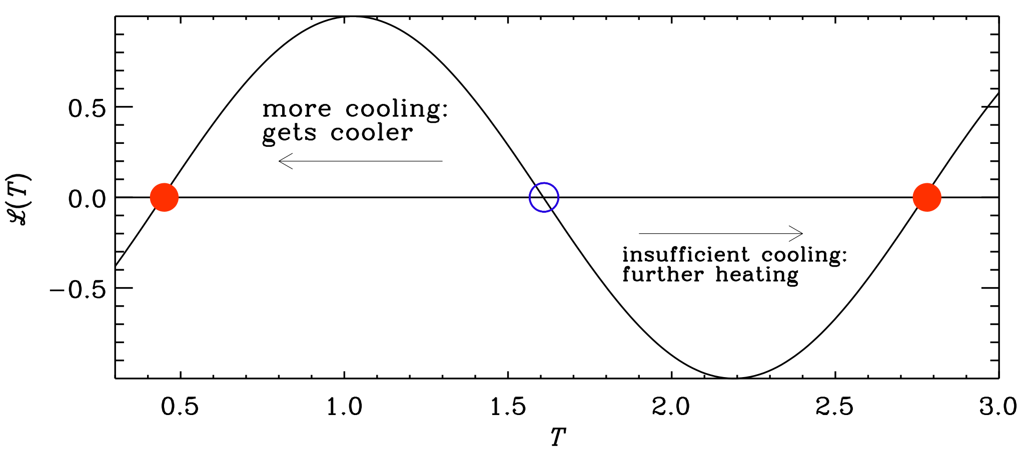
(a) Qualitative sketch showing a possible form of $\mathcal{L}(T)$ that leads to an instability:



(b) Fixed points correspond to $\mathcal{L}(T) = 0$. They are marked with open and closed symbols. When $\mathcal{L}(T) > 0$, the gas will cool, so T will sink. This is indicated by an arrow pointing left. Therefore, the fixed point in the middle (blue open symbol) is unstable and the one to the left is stable. Likewise, when $\mathcal{L}(T) < 0$, the gas will heat, so again the fixed point in the middle (blue open symbol) is unstable and the one to the right is stable.

(c) If the gas cools, its pressure decreases. This leads to compression, so the density increases. This enhances the cooling effect, which is proportional $\rho^2\Lambda$, and makes the gas more easily unstable. Conversely, if the gas heats, it expands, the density decreases, decreasing its ability to cool, making it again more easily unstable. The suggests that the isochoric stabiluty criterion may not be as stringent as if density adjustment is allowed.

2. **Stability with self-gravity.** Consider the linearized one-dimensional continuity, Euler, and Poisson equations in the form

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_{1x}}{\partial x},$$

$$\frac{\partial u_{1x}}{\partial t} = -\frac{c_\mathrm{s}^2}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\partial \Phi_1}{\partial x},$$

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G \rho_1,$$

where subscripts 1 indicate small perturbations and c_s and ρ_0 are assumed constant.

(a) Assume that all perturbed quantities (subscript 1) are proportional to $e^{\mathrm{i}(kx-\omega t)}$ and write the resulting equations in matrix form.

(b) Show that the dispersion relation is

$$\omega^2 = c_\mathrm{s}^2 k^2 - 4\pi G \rho_0,$$

and state for which values of k^2 the system is unstable.

(c) Define

$$D = c_\mathrm{s}^2 - 4\pi G \rho_0 / k^2,$$

and verify that

$$\begin{pmatrix} \rho_1 \\ u_{1x} \\ \Phi_1 \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \pm\sqrt{D} \\ -4\pi G \rho_0 / k^2 \end{pmatrix} \cos k(x \mp \sqrt{D} t)$$

is an eigenfunction for $D > 0$.

(d) Give an eigenfunction for $D < 0$.

(e) Give an example where the above equations are relevant and describe in a few words what happens.

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(a) Assume that all perturbed quantities are proportional to $e^{\mathrm{i}(kx-\omega t)}$, and moving all terms to the left, we have

$$\begin{pmatrix} -\mathrm{i}\omega & \mathrm{i}k & 0 \\ \mathrm{i}k c_\mathrm{s}^2 & -\mathrm{i}\omega & +\mathrm{i}k \\ 4\pi G \rho_0 & 0 & k^2 \end{pmatrix} \begin{pmatrix} \hat{\rho}_1/\rho_0 \\ \hat{u}_{1x} \\ \hat{\phi}_1 \end{pmatrix} = 0. \tag{2}$$

(b) Computing the determinant of the matrix yields

$$-\omega^2 k^2 - 4\pi G \rho_0 k^2 + c_\mathrm{s}^2 k^4 = 0, \tag{3}$$

or

$$k^2(\omega^2 - c_\mathrm{s}^2 k^2 + 4\pi G \rho_0) = 0, \tag{4}$$

so the dispersion relation reads

$$\omega^2 = c_\mathrm{s}^2 k^2 - 4\pi G \rho_0 \tag{5}$$

The system is unstable when $\omega^2 < 0$, i.e., in the range $0 \le k^2 < 4\pi G \rho_0 / c_\mathrm{s}^2$.

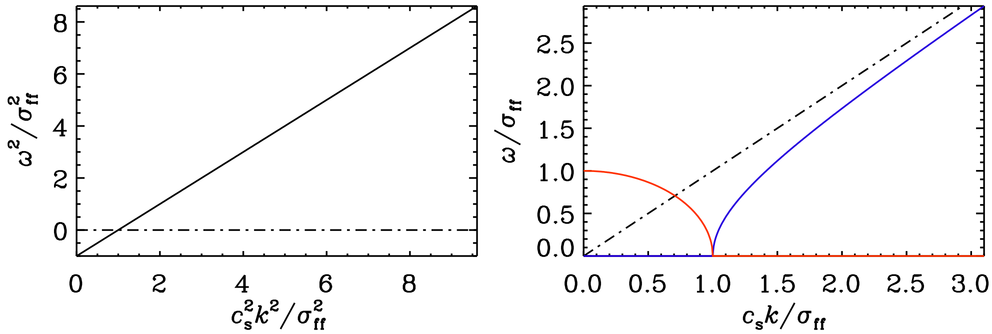

Figure 1: Dispersion relation showing $\omega^2(c_s^2k^2)/\sigma_{\rm ff}^2$ (left) and $\omega(c_sk)/\sigma_{\rm ff}$ (right), where $\sigma_{\rm ff}^2 = 4\pi G\rho_0$ has been introduced. In the right-hand panel, imaginary (real) parts are plotted in red (blue). On the left, the dispersion relation for sound waves is plotted as a dash-dotted line.

(c) To verify this, we just insert this into the original equations. We begin with

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_{1x}}{\partial x},$$

and find

$$\frac{\partial \rho_1}{\partial t} = \pm\rho_0 k\sqrt{D}\sin k(x \mp \sqrt{D}t)$$

on the left and

$$-\rho_0 \frac{\partial u_{1x}}{\partial x} = \pm\rho_0 k\sqrt{D}\sin k(x \mp \sqrt{D}t)$$

on the right. Next, we consider

$$\frac{\partial u_{1x}}{\partial t} = -\frac{c_s^2}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\partial \Phi_1}{\partial x},$$

and find

$$\frac{\partial u_{1x}}{\partial t} = Dk\sin k(x \mp \sqrt{D}t)$$

on the left and

$$-\frac{c_s^2}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\partial \Phi_1}{\partial x} = c_s^2 k\sin k(x \mp \sqrt{D}t) - 4\pi G\rho_0/k\sin k(x \mp \sqrt{D}t)$$

on the right. This would mean

$$Dk^2 = c_s^2k^2 - 4\pi G\rho_0$$

which is indeed obeyed given the definition of D. Finally, we consider

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G\rho_1,$$

and find

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G\rho_0\cos k(x \mp \sqrt{D}t)$$

on the left and

$$4\pi G\rho_1 = 4\pi G\rho_0\cos k(x \mp \sqrt{D}t)$$

which is also ok.

(d) Let's guess (and then check [and then modify correspondingly]):

$$\begin{pmatrix} \rho_1 \\ u_{1x} \\ \Phi_1 \end{pmatrix} = \begin{pmatrix} \rho_0 \cos kx \\ \mp\sqrt{D} \sin kx \\ -(4\pi G\rho_0/k^2)\, \cos kx \end{pmatrix} e^{\mp\sqrt{D}t}$$

To verify this, we just insert this into the original equations. We begin with

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial u_{1x}}{\partial x},$$

and find

$$\frac{\partial \rho_1}{\partial t} = \mp\rho_0 k\sqrt{D} \cos kx\, e^{\mp\sqrt{D}t}$$

on the left and

$$-\rho_0 \frac{\partial u_{1x}}{\partial x} = \mp\rho_0 k\sqrt{D} \cos kx\, e^{\mp\sqrt{D}t}$$

on the right. Next, we consider

$$\frac{\partial u_{1x}}{\partial t} = -\frac{c_{\mathrm{s}}^2}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\partial \Phi_1}{\partial x},$$

and find

$$\frac{\partial u_{1x}}{\partial t} = +Dk \sin kx\, e^{\mp\sqrt{D}t}$$

on the left and

$$-\frac{c_{\mathrm{s}}^2}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\partial \Phi_1}{\partial x} = c_{\mathrm{s}}^2 k \sin kx\, e^{\mp\sqrt{D}t} - (4\pi G\rho_0/k) \sin kx\, e^{\mp\sqrt{D}t}$$

on the right. This would mean

$$+Dk^2 = c_{\mathrm{s}}^2 k^2 - 4\pi G\rho_0$$

which is indeed obeyed given the definition of D. Finally, we consider

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G\rho_1,$$

and find

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G\rho_0 \cos kx\, e^{\mp\sqrt{D}t}$$

on the left and

$$4\pi G\rho_1 = 4\pi G\rho_0 \cos kx\, e^{\mp\sqrt{D}t}$$

which is also ok.

(e) The interstellar medium seems to be the place where these equations are relevant (the isothermal equation of state is commonly used and there is no rotation, unlike to protosolar nebula). On short enough length scales, sound waves work as usual, but on larger length scales, they are no longer propagating and become unstable. This can lead to star formation, especially when the temperatures are low and the sound speed small. In practice, rotation sooner or later will play a role.

3. **The f-mode.** Consider the eigenvalue problem for p- and f-mode oscillations in a polytropic atmosphere,

$$\omega^2\hat{\Phi} = c_\mathrm{s}^2 k_\perp^2\,\hat{\Phi} + g\frac{\partial\hat{\Phi}}{\partial z} - c_\mathrm{s}^2\frac{\partial^2\hat{\Phi}}{\partial^2 z}, \tag{6}$$

where ω is the eigenfrequency, $c_\mathrm{s}(z)$ is the speed of sound, $k_\perp$ is the horizontal wavenumber, and $\hat{\Phi}(z)$ is the eigenfunction, which is related to the displacement $\boldsymbol{\xi}(\boldsymbol{x}_\perp, z, t)$ for the velocity $\boldsymbol{u} = \mathrm{D}\boldsymbol{\xi}/\mathrm{D}t$ via $\boldsymbol{\xi} = \boldsymbol{\nabla}\Phi$ with $\Phi(\boldsymbol{x}_\perp, z, t) = \hat{\Phi}(z)\cos(\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp)\,\cos(\omega t)$. Furthermore, we have

$$\mathrm{D}\ln P/\mathrm{D}t = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{7}$$

which followed from $\mathrm{D}\ln\rho/\mathrm{D}t = -\boldsymbol{\nabla}\cdot\boldsymbol{u}$ and $\mathrm{D}S/\mathrm{D}t = 0$.

(a) Show that the assumption of zero pressure fluctuations at the surface is equivalent to $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$, i.e., $\nabla^2\Phi = 0$.

(b) Show that the surface condition $\nabla^2\Phi = 0$ corresponds to $\hat{\Phi} = \hat{\Phi}_0\,\exp(-k|z|)$.

(c) Show that one particular solution (corresponding to the f-mode) is given by

$$\omega^2 = gk_\perp. \tag{8}$$

. . . .

(a) Inserting $\boldsymbol{u} = \mathrm{D}\boldsymbol{\xi}/\mathrm{D}t$ into Eq. (7) yields

$$\mathrm{D}\ln P/\mathrm{D}t = -\gamma\boldsymbol{\nabla}\cdot\mathrm{D}\boldsymbol{\xi}/\mathrm{D}t,$$

Integration leads to

$$\ln P = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$$

Linearization yields $\ln P = \ln(P_0 + P_1) = \ln[P_0(1 + P_1/P_0)] = \ln P_0 + \ln(1 + P_1/P_0) = \ln P_0 + P_1/P_0$. Zero pressure fluctuations imply therefore $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0$ for the fluctuations $\boldsymbol{\xi}_1$.

(b) From $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0$ we have $\nabla^2\Phi_1 = 0$ on the surface. This means

$$\Phi'' - k^2\Phi = 0,$$

which has the solution $\Phi = \Phi_0\,e^{\pm kz}$. For this solution to be physical, it must not blow up at infinity, so we have $\Phi = \Phi_0\,e^{-|kz|}$.

(c) For $z < 0$ (below the surface), we have $\Phi = \Phi_0\,e^{k_\perp z}$. Inserting this into Eq. (6) yields

$$\omega^2\hat{\Phi}_0 = gk_\perp\hat{\Phi}_0$$

or $\omega^2 = gk_\perp$.

4. **Magnetic energy equation.** Assume that the magnetic field, $\boldsymbol{B}$, is governed by the equations

$$\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{E} = 0, \quad \boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}/\mu_0, \quad \boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B},$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ the current density, $\boldsymbol{v}$ the velocity, and μ_0 the permeability.

(a) Show that

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{E} = 0.$$

(b) Write $\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \epsilon_{ijk}\partial_i(E_j B_k)$ and show, using the product rule, that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}.$$

[Remember that $\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{jik}$.]

(c) Use this relation to show that

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) + \boldsymbol{J} \cdot \boldsymbol{E} = 0.$$

(d) Show that the energy equation can be written in the form

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) + \boldsymbol{v} \cdot (\boldsymbol{J} \times \boldsymbol{B}) = 0.$$