# Handout 0: Basic equations

## 1 Continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

It can be written in various other forms, such as $\mathrm{D} \ln \rho / \mathrm{D}t = -\boldsymbol{\nabla} \cdot \boldsymbol{u}$.

## 2 Navier-Stokes equation

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi + \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \tag{2}$$

where $\Phi$ is the gravitational potential, which obeys $\nabla^2 \Phi = 4\pi G \rho$. The solution for a sphere, for example, is $\Phi = -GM/r$, while for a plane layer it is $\Phi = zg$, up to an additive constant.

For a monatomic gas, the stress tensor is $\boldsymbol{\tau} = 2\rho\nu \mathsf{S}$ with $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij} \boldsymbol{\nabla} \cdot \boldsymbol{u}$. In that case, $\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \rho\nu(\nabla^2 \boldsymbol{u} + \frac{1}{3} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u} + 2\mathsf{S}\boldsymbol{\nabla} \ln \rho\nu)$.

- Note that the $\boldsymbol{\nabla}\Phi$ term comes with a minus and an additional $\rho$ factor.
- All the other terms come with no $\rho$ factor.
- It is sometimes useful to note that $\rho^{-1}\boldsymbol{\nabla} P = \boldsymbol{\nabla} h - T\boldsymbol{\nabla} S$, where $h$ is the specific enthalpy. For an ideal gas we have $h = c_{\mathrm{p}} T$.

## 3 Energy equation

$$\rho T \frac{\mathrm{D}S}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{F}_{\mathrm{rad}} + \boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{J}^2/\sigma + \text{ nucl. fusion} \tag{3}$$

The $\boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{u}$ is supposed to mean $\tau_{ij} u_{i,j}$. This is the viscous heating term and it is *positive definite*. In the homework, we are supposed to find that for $\boldsymbol{\tau} = 2\rho\nu\mathsf{S}$ we have $\boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{u} = 2\rho\nu\mathsf{S}^2$, which is manifestly positive definite. In the optically thick case, the radiative flux is $\boldsymbol{F}_{\mathrm{rad}} = -K\boldsymbol{\nabla} T$. The electric conductivity $\sigma$ can also be written as $1/\sigma = \mu_0 \eta$, where $\mu_0$ is the vacuum permeability and $\eta$ the magnetic diffusivity.

- In the strictly isentropic case ($S = \mathrm{const}$), we have $P \propto \rho^\gamma$. However, viscous heating can usually not be neglected!
- The lhs of Equation (3) can also be written as $\rho \mathrm{D}e/\mathrm{D}t + P\boldsymbol{\nabla} \cdot \boldsymbol{u}$.
- For an ideal gas, we have $e = c_{\mathrm{v}} T$ and can write the lhs also as $\rho c_{\mathrm{v}} \mathrm{D}T/\mathrm{D}t + P\boldsymbol{\nabla} \cdot \boldsymbol{u}$.
- Again for an ideal gas, we can write the lhs as $\rho c_{\mathrm{p}} \mathrm{D}T/\mathrm{D}t - \mathrm{D}P/\mathrm{D}t$.

Recall that $\mathcal{R}/\mu = c_{\mathrm{p}} - c_{\mathrm{v}}$, where $\mathcal{R} = k_{\mathrm{B}}/m_{\mathrm{u}}$ is the universal gas constant[1] and $\mu$ is the mean molecular weight (=1 for neutral hydrogen, $\approx 1.2$ for a neutral hydrogen-helium mixture, 0.6 for an ionized hydrogen-helium mixture).

## 4 Induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{J}/\sigma) \tag{4}$$

[1]Here, $k_{\mathrm{B}} = 1.3806505 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$ is the Boltzmann constant, $m_{\mathrm{u}} = 1.66053886^{-24}\,\mathrm{g}$ is the atomic mass unit [0.993 times the proton mass], and so $\mathcal{R} = 8.31447 \times 10^7\,\mathrm{erg\,g^{-1}\,K^{-1}}$. In astrophysics, $\mu$ is dimensionless rather than in g/mol.