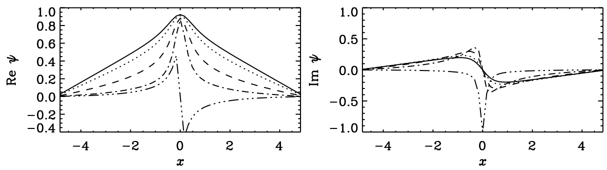
Handout 10: Eigenvectors as *vectors*

The purpose of this handout is to give more substance to the notion of eigenfunctions and to plot them in real space as a function of x and y, as done in Figure 1. At the bottom of Problems Set 3 (September 22), the real and imaginary parts of the eigenfunction were plotted like so:



Here, $\mathrm{Re}\hat{\psi}(x)$ and $\mathrm{Im}\hat{\psi}(x)$ were obtained as *eigenvectors* of an $n \times n$ matrix. Let us recall that in Handout 7, the velocity was written in the form

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{z}}), \tag{1}$$

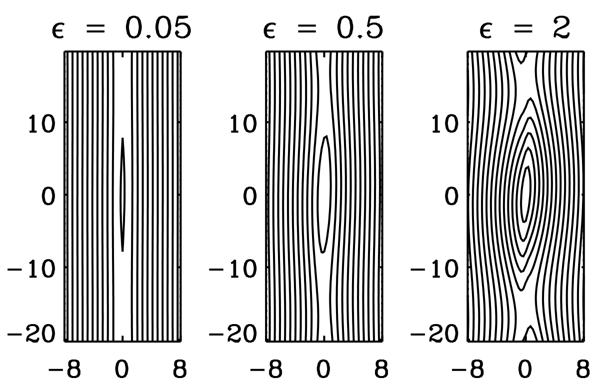
and the streamfunction ψ was written as $\psi(x, y, t) = \hat{\psi}(z)\, e^{\mathrm{i}ky+\sigma t}$. Correspondingly, we have $\hat{u}_x$ and $\hat{u}_y$, which are given by

$$\hat{u}_x = \mathrm{i}k\hat{\psi}, \quad \hat{u}_y = -\hat{\psi}'. \tag{2}$$

This is not very intuitive and at the end of the day we want to see how the flow really looks like. This can be done computing $\psi(x, y) = \hat{\psi}(z)\, e^{\mathrm{i}ky}$ and plotting it as contours. These contours are indeed the streamlines of the perturbed flow.

Alternatively, we can plot $\psi(x, y)$ as "filled" contours (color scale representation) together with the $\boldsymbol{u}$ vectors. This is shown in Figure 1, where

$$u_x = \mathrm{Re}\left(\mathrm{i}k\hat{\psi}\, e^{\mathrm{i}ky}\right), \quad u_y = \mathrm{Re}\left(-\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}x}\, e^{\mathrm{i}ky}\right). \tag{3}$$



We must remember, however, that this is only the *perturbation* and that ψ was actually only ψ_1, so

$$\psi(x, y) = \psi_0(x) + \epsilon\psi_1(x, y), \tag{4}$$

where $\psi_0(x) = U_0 \ln\cosh x$ for $U = U_0 \tanh x$ with U_0 being the flow amplitude and ϵ is an *assumed* perturbation amplitude; see the three panels above.

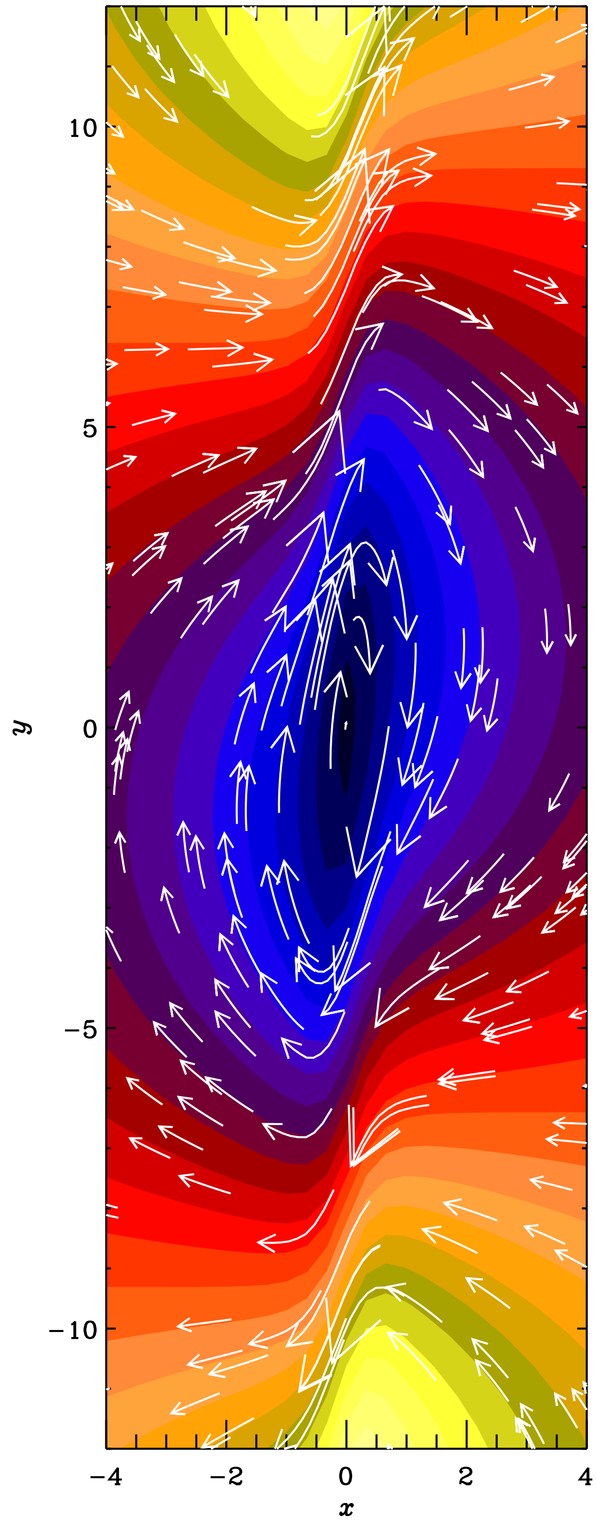


Figure 1: Eigenfunction ψ for Rayleigh's instability equation plotted as a function of x and y.

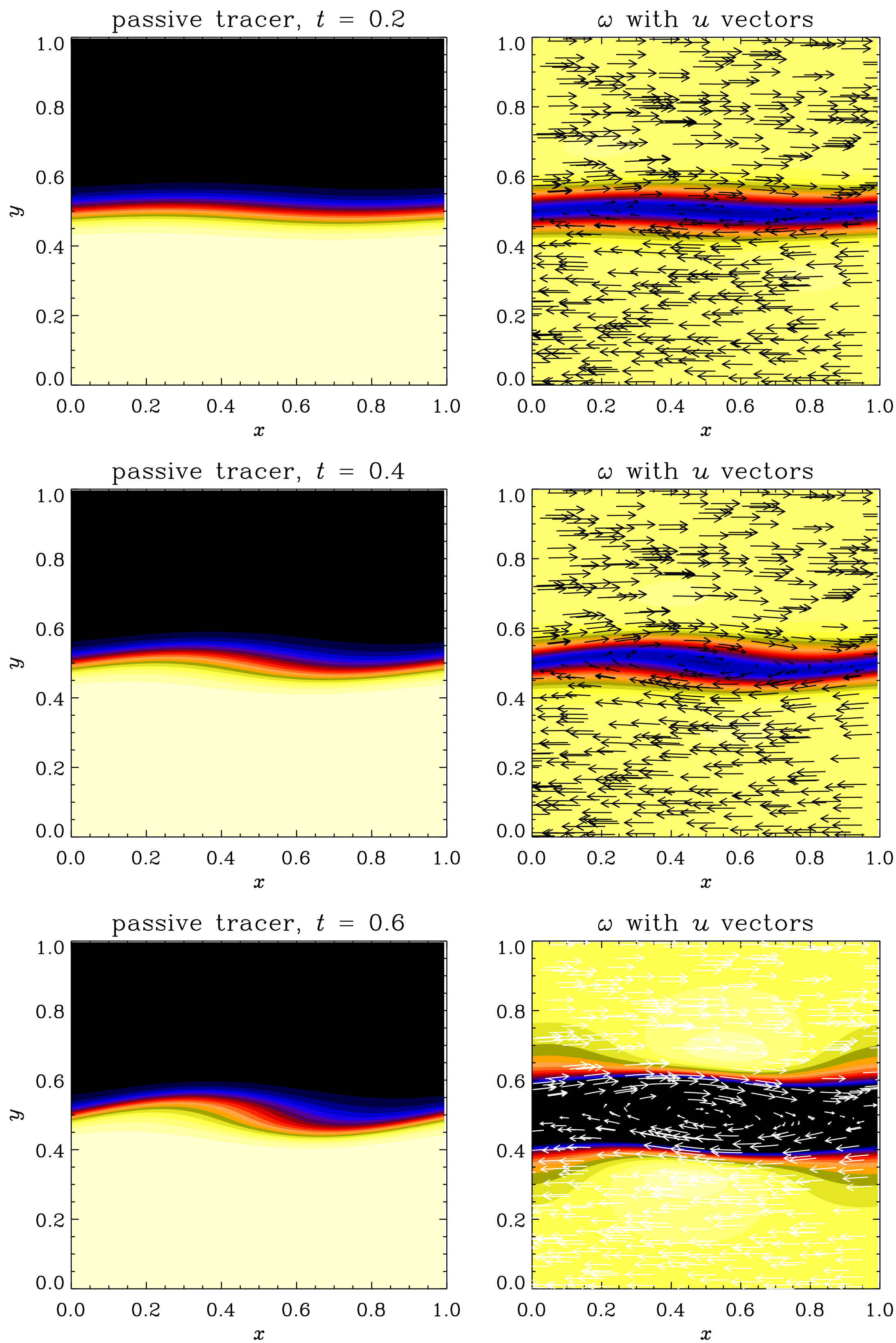


Figure 2: Numerical solutions obtained with the Pencil Code.

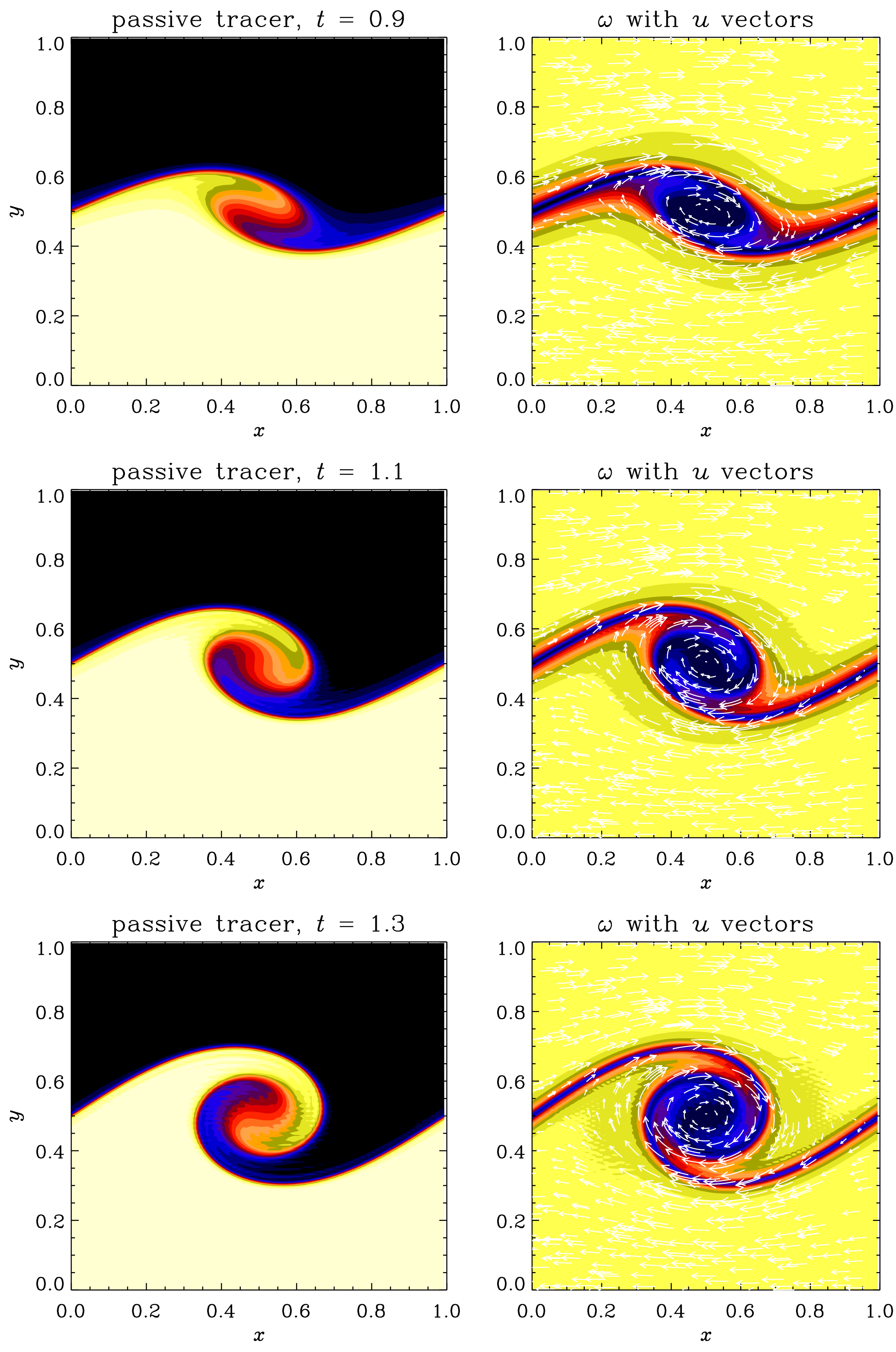


Figure 3: Similar to Figure **??**, but for later times.

Nonlinear solutions

We have seen the linear solutions[1] in the figure above still look somewhat odd. Thus is because ϵ should really be rather small and 0.2 is too big. On the other hand, for $\epsilon = 0.05$, there is not much to see.

To solve the fully nonlinear problem, we can use a hydrodynamics code such as ATHENA[2], DEDALUS[3], or the PENCIL CODE[4]. The Rayleigh instability problem is a limiting case ($\nu \to 0$) of the Kelvin-Helmholtz instability, which is used as a common test problem for numerical codes. It has been used last Summer during the Bootcamp for Computational Fluid Dynamics`http://www.nordita.org/~brandenb/teach/PencilCode/LCDworkshop2016/` at the Laboratory for Computational Dynamics (LCD), which is upstairs in the third floor. See Lecoanet et al. (2016) for details.

References

Lecoanet, D., McCourt, M., Quataert, E., Burns, K. J., Vasil, G. M., Oishi, J. S., Brown, B. P., Stone, J. M., & O'Leary, R. M., "A validated non-linear Kelvin-Helmholtz benchmark for numerical hydrodynamics," *Month. Not. Roy. Astron. Soc.* **455**, 4274-4288 (2016).

$Header: /var/cvs/brandenb/tex/teach/ASTR_5410/10_Eigenvectors/notes.tex,v 1.2 2016/09/29 12:32:43 brandenb Exp $

[1] `http://lcd-www.colorado.edu/~axbr9098/teach/ASTR_5410/lectures/10_Eigenvectors/idl/`
[2] `http://www.astro.princeton.edu/~jstone/athena.html`
[3] `http://dedalus-project.org/`
[4] `http://pencil-code.nordita.org/`