Handout 11: p-modes for the isentropic case

1 Derivation for isentropic case

The continuity equations is $\mathrm{D}\ln\rho/\mathrm{D}t = -\boldsymbol{\nabla}\cdot\boldsymbol{u}$, and from $\mathrm{D}s/\mathrm{D}t = 0$ we obtain $\mathrm{D}\ln p/\mathrm{D}t = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{u}$. Our linearized equations are then

$$\dot{\rho_1} + \boldsymbol{u}_1\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = 0, \tag{1}$$

$$\dot{p_1} + \boldsymbol{u}_1\cdot\boldsymbol{\nabla}p_0 + \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = 0, \tag{2}$$

$$\rho_0\dot{\boldsymbol{u}}_1 + \boldsymbol{\nabla}p_1 + \rho_1 g\hat{\boldsymbol{z}} = 0. \tag{3}$$

Assume $\dot{\boldsymbol{\xi}} = \boldsymbol{u}$ and integrate the continuity equation in time, i.e.,

$$\rho_1 + \boldsymbol{\xi}_1\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0, \tag{4}$$

$$p_1 + \boldsymbol{\xi}_1\cdot\boldsymbol{\nabla}p_0 + \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0, \tag{5}$$

$$\rho_0\ddot{\boldsymbol{\xi}}_1 + \boldsymbol{\nabla}p_1 + \rho_1 g\hat{\boldsymbol{z}} = 0. \tag{6}$$

Divide by ρ_0, use $\boldsymbol{\nabla}p_0 = -\rho_0 g\hat{\boldsymbol{z}}$, $\gamma p_0 = c_\mathrm{s}^2\rho_0$, replace $g\hat{\boldsymbol{z}} = c_\mathrm{s}^2\hat{\boldsymbol{z}}/H_\rho = -c_\mathrm{s}^2\boldsymbol{\nabla}\ln\rho_0$, and insert the above expressions for p_1 and ρ_1, so

$$\rho_1 + \boldsymbol{\xi}_1\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0, \tag{7}$$

$$p_1 - \xi_{1z}\rho_0 g + c_\mathrm{s}^2\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1 = 0, \tag{8}$$

$$\ddot{\boldsymbol{\xi}}_1 + \frac{1}{\rho_0}\boldsymbol{\nabla}p_1 + \frac{\rho_1}{\rho_0}g\hat{\boldsymbol{z}} = 0. \tag{9}$$

Insert p_1 and ρ_1 into the momentum equation, so

$$\ddot{\boldsymbol{\xi}}_1 + \frac{1}{\rho_0}\boldsymbol{\nabla}\left[\rho_0(\xi_{1z}g - c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)\right] - (\boldsymbol{\xi}_1\cdot\boldsymbol{\nabla}\ln\rho_0 + \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)g\hat{\boldsymbol{z}} = 0, \tag{10}$$

or

$$\ddot{\boldsymbol{\xi}}_1 + \frac{1}{\rho_0}\boldsymbol{\nabla}\left[\rho_0(\xi_{1z}g - c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)\right] + (\xi_{1z}/H_\rho - \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)g\hat{\boldsymbol{z}} = 0, \tag{11}$$

or, using the fact that $gH_\rho = c_\mathrm{s}^2$, we get

$$\ddot{\boldsymbol{\xi}}_1 + \frac{1}{\rho_0}\boldsymbol{\nabla}\left[\rho_0(\xi_{1z}g - c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)\right] + (\xi_{1z}g - c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1)\hat{\boldsymbol{z}}/H_\rho = 0, \tag{12}$$

so

$$\ddot{\boldsymbol{\xi}}_1 + \boldsymbol{\nabla}\left(\xi_{1z}g - c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_1\right) = 0. \tag{13}$$

Assuming that the displacement is a potential field, i.e., $\boldsymbol{\xi}_1 = \boldsymbol{\nabla}\Phi$, we have (Bogdan & Cally, 1995)

$$\ddot{\Phi} + g\partial\Phi/\partial z - c_\mathrm{s}^2\nabla^2\Phi = 0. \tag{14}$$

This leads to an eigenvalue problem for eigenvalue ω^2,

$$\omega^2\Phi = c_\mathrm{s}^2k^2\Phi + g\frac{\partial\Phi}{\partial z} - c_\mathrm{s}^2\frac{\partial^2\Phi}{\partial^2 z}. \tag{15}$$

At the bottom, we assume $\Phi = 0$, which means that the last data point has to be omitted from the matrix. At the top, we require $(k^2 - \mathrm{d}^2/\mathrm{d}z^2)\Phi = 0$, which means that we replace the eigenvalue problem on the boundary by

$$\omega^2\Phi = g\frac{\partial\Phi}{\partial z} \tag{16}$$

Note that Equation (15) is valid even in the non-isothermal case. In Figure 1, we compare solutions for the isothermal and non-isothermal ($n = 3/2$) cases. Note that in the latter case the p-modes bend down and the difference between qualitative difference between the f- and p-modes diminishes.

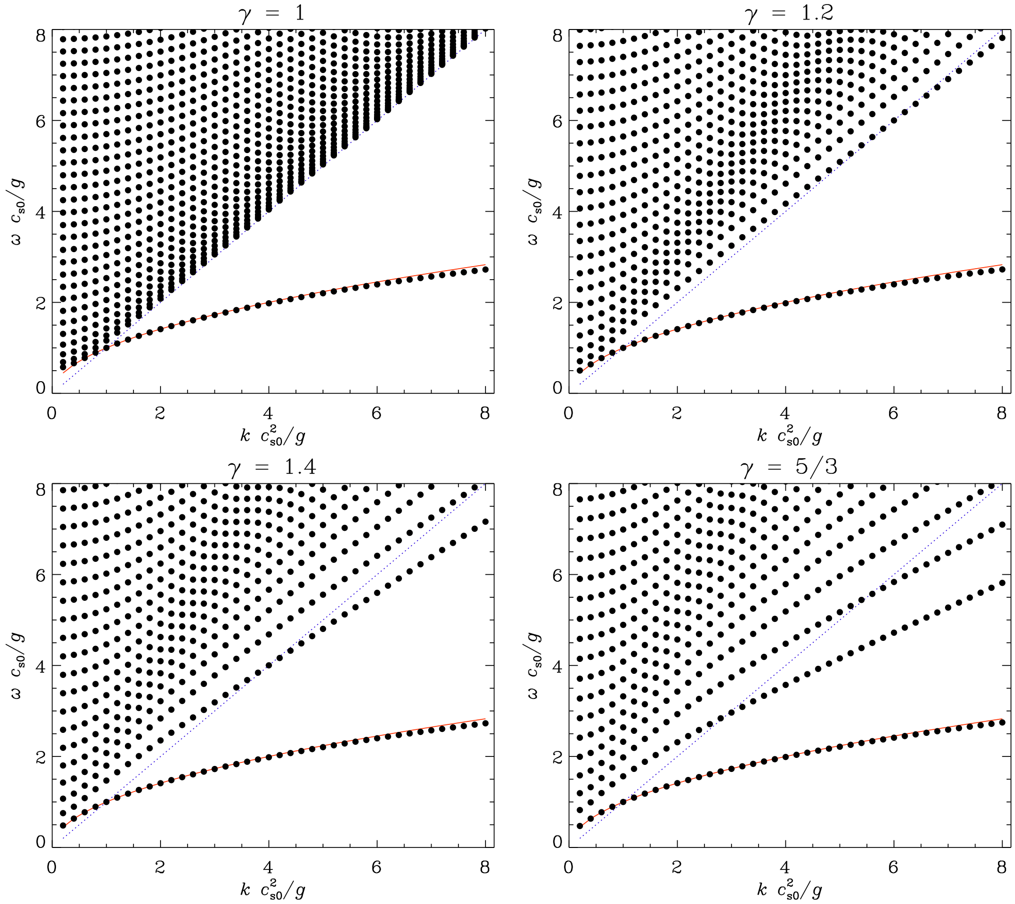

Figure 1: $k\omega$ diagram for isothermal case (left) and various isentropic cases.

2 Reduction to the isothermal case

It is instructive to see how to obtain the dispersion relation for the isothermal case from Equation (15). Replacing $\partial_z \to \mathrm{i}k_z$, we have

$$\omega^2 = c_\mathrm{s}^2 k^2 + \mathrm{i}k_z g + c_\mathrm{s}^2 k_z^2 = 0, \tag{17}$$

Split $k_z = k_z' + \mathrm{i}k_z''$ into real and imaginary parts, so we get

$$\omega^2 = c_\mathrm{s}^2 k^2 + (\mathrm{i}k_z' - k_z'')g + c_\mathrm{s}^2(k_z'^2 - k_z''^2 + 2\mathrm{i}k_z' 2k_z'') = 0, \tag{18}$$

and solve for real and imaginary parts:

$$\omega^2 = c_\mathrm{s}^2 k^2 - k_z'' g + c_\mathrm{s}^2(k_z'^2 - k_z''^2) = 0, \tag{19}$$

$$g k_z' + 2k_z' 2k_z'' c_\mathrm{s}^2 = 0, \tag{20}$$

So now replace $k_z'' = -g/2c_\mathrm{s}^2$ into Equation (19) and get

$$\omega^2 = c_\mathrm{s}^2 k^2 + g^2/2c_\mathrm{s}^2 + c_\mathrm{s}^2(k_z'^2 - g^2/4c_\mathrm{s}^4) = 0, \tag{21}$$

so

$$\omega^2 = c_\mathrm{s}^2 k^2 + g^2/4c_\mathrm{s}^2 + c_\mathrm{s}^2 k_z'^2 = 0. \tag{22}$$

References

Bogdan, T. J., & Cally, P. S., "Jacket Modes: Solar Acoustic Oscillations Confined to Regions Surrounding Sunspots and Plage," *Astrophys. J.* **453**, 919-928 (1995).

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