Handout 12: Acoustic cavity

In Handout 11 we derived the following eigenvalue problem for eigenvalue ω^2,

$$(\omega^2 - c_\mathrm{s}^2 k^2)\,\Phi - g\Phi' + c_\mathrm{s}^2\Phi'' = 0 \tag{1}$$

with boundary condition $\Phi = 0$ at the bottom and $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$, or $(k^2 - \mathrm{d}^2/\mathrm{d}z^2)\Phi = 0$, at the top. We assume a polytropic stratification with

$$c_\mathrm{s}^2 = c_\mathrm{s0}^2 - (\gamma - 1)\,gz. \tag{2}$$

We can bring Equation (1) into Sturm-Liouville form by substituting

$$\Phi(z) = f(z)\,\phi(z), \tag{3}$$

so

$$\Phi' = f'\phi + f\phi' \tag{4}$$

and

$$\Phi'' = f''\phi + 2f'\phi' + f\phi''. \tag{5}$$

Thus, we have

$$(\omega^2 - c_\mathrm{s}^2 k^2)\,f\phi - g(f'\phi + f\phi') + c_\mathrm{s}^2(f''\phi + 2f'\phi' + f\phi'') = 0 \tag{6}$$

We now choose f such that the ϕ' term vanishes, i.e.,

$$-gf + 2c_\mathrm{s}^2 f' = 0, \tag{7}$$

so that

$$\ln f = \tfrac{1}{2}\int \frac{g}{c_\mathrm{s}^2}\,\mathrm{d}z. \tag{8}$$

Using Equation (2), we have $\mathrm{d}c_\mathrm{s}^2/\mathrm{d}z = -(\gamma-1)g$, so

$$\ln f = \frac{1}{\gamma-1}\int\frac{\mathrm{d}c_\mathrm{s}^2}{2c_\mathrm{s}^2} = \frac{1}{\gamma-1}\int\frac{\mathrm{d}c_\mathrm{s}}{c_\mathrm{s}} = \frac{\ln c_\mathrm{s}}{\gamma-1} \tag{9}$$

or simply $f = c_\mathrm{s}^{1/(\gamma-1)} = c_\mathrm{s}^n$, where $n = 1/(\gamma-1)$ is the polytropic index.

Having eliminated the ϕ' term, our differential equation takes the form

$$(\omega^2 - c_\mathrm{s}^2 k^2)\,f\phi - gf'\phi + c_\mathrm{s}^2(f''\phi + f\phi'') = 0, \tag{10}$$

or

$$\left(\omega^2/c_\mathrm{s}^2 - k^2 + f''/f - gf'/f\right)\phi + \phi'' = 0, \tag{11}$$

where $f'/f = nc_\mathrm{s}'/c_\mathrm{s}$ and $f''/f = n(n-1)c_\mathrm{s}'^2/c_\mathrm{s}^2 + nc_\mathrm{s}''/c_\mathrm{s}$. Let us now introduce the abbreviation $K(z)^2 = \omega^2/c_\mathrm{s}^2 - k^2 + f''/f - gf'/f$, so we have

$$K^2\phi + \phi'' = 0. \tag{12}$$

We now proceed by solving Equation (12) using WKB. If K was constant, the solution would be of the form $\phi = e^{\mathrm{i}K}$. Thus, we make the ansatz

$$\phi(z) = A(z)\,e^{\mathrm{i}\varphi(z)}, \tag{13}$$

where $A(z)$ is a slowly varying function. Let us first calculate the first and second derivatives, so

$$\phi' = (A' + \mathrm{i}A\varphi')\,e^{\mathrm{i}\varphi}, \tag{14}$$

$$\phi'' = (A'' + \mathrm{i}A'\varphi' + \mathrm{i}A\varphi'')\,e^{\mathrm{i}\varphi} + (A' + \mathrm{i}A\varphi')\,\mathrm{i}\varphi'\,e^{\mathrm{i}\varphi}, \tag{15}$$

i.e.,

$$\phi'' = \left[(A'' - A\varphi'^2) + \mathrm{i}(2A'\varphi' + A\varphi'') \right] e^{\mathrm{i}\varphi} \tag{16}$$

Inserting this into Equation (12), canceling the $e^{\mathrm{i}\varphi}$ term everywhere, and demanding that real and imaginary parts vanish separately, we have

$$(K^2 - \varphi'^2)A + A'' = 0, \quad 2A'\varphi' + A\varphi'' = 0. \tag{17}$$

We now make use of the fact that A is slowly varying, so we drop A'' in favor of the $(K^2 - \varphi'^2)A$ term, which then implies that

$$\varphi' = \pm K \tag{18}$$

and therefore

$$\varphi(z) = \pm \int^z K(z')\,\mathrm{d}z' \tag{19}$$

Integrating the second part of Equation (17) yields

$$\mathrm{d}\ln A/\mathrm{d}z = -\tfrac{1}{2}\mathrm{d}\ln\varphi'/\mathrm{d}z = -\tfrac{1}{2}\mathrm{d}\ln K/\mathrm{d}z, \tag{20}$$

so

$$A = A_0 K^{-1/2}. \tag{21}$$

and therefore Equation (22) yields

$$\phi(z) = \frac{A_0}{\sqrt{K}}\, e^{\pm \mathrm{i} \int K(z')\,\mathrm{d}z'} \tag{22}$$

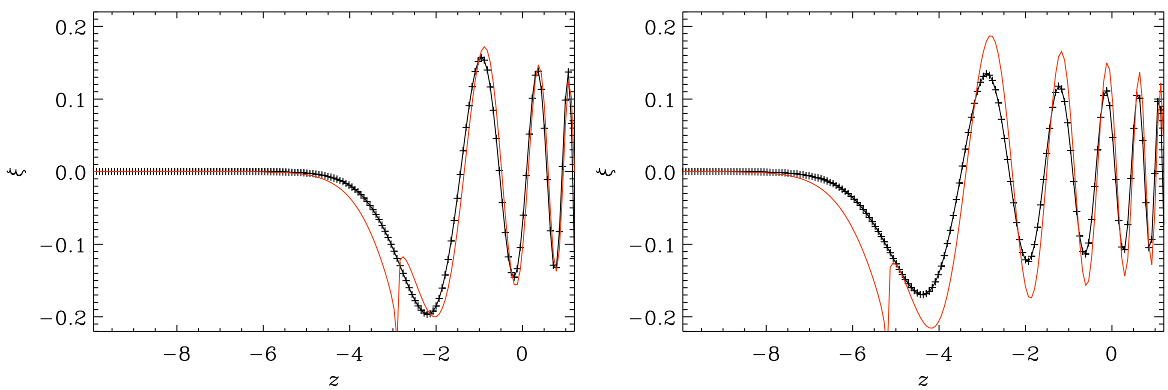

Figure 1: Eigenfunctions from WKB (red) and the eigenvalue solver.

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