Handout 12b: Ray tracing

Ray tracing means that one solves the eikonal equation, which is a nonlinear partial differential equation that emerges in problems of wave propagation, when the wave equation is approximated using the WKB theory. Here one follows a ray path $\boldsymbol{r} = \boldsymbol{r}(t)$ and its direction $\boldsymbol{k} = \boldsymbol{k}(t)$ by solving the eikonal equations,

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}_\mathrm{g}, \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\boldsymbol{\nabla}\omega, \tag{2}$$

where $\boldsymbol{v}_\mathrm{g} = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega$ is the group velocity and $\omega = \omega(\boldsymbol{r}, \boldsymbol{k})$ is the known dispersion relation of the wave.

1 Sound waves

Take as an example a polytropic layer with $c_\mathrm{s} = (z_0 - z)g$, so

$$\boldsymbol{\nabla}\omega = -\frac{gk}{2c_\mathrm{s}}\boldsymbol{z} \tag{3}$$

but this we do numerically. Without magnetic fields, the Doppler-shifted frequency is $\omega = \boldsymbol{U}\cdot\boldsymbol{k} + \sqrt{c_\mathrm{s}^2\boldsymbol{k}^2}$, so the group velocity is $\boldsymbol{v}_g = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega = \boldsymbol{U} + c_\mathrm{s}\hat{\boldsymbol{k}}$.

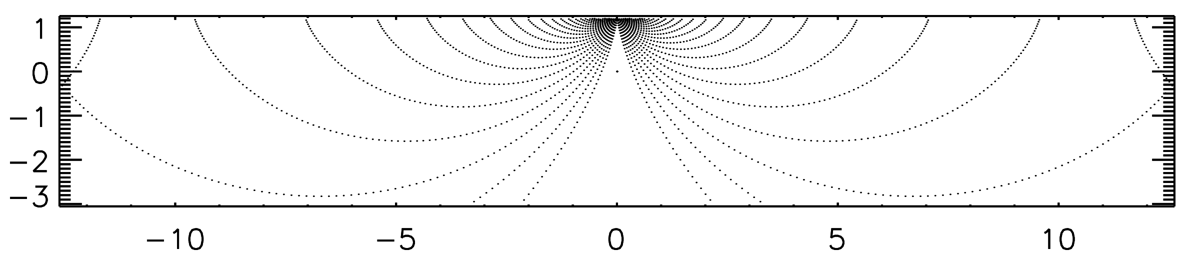

Figure 1: Rays launched from one point at the surface.

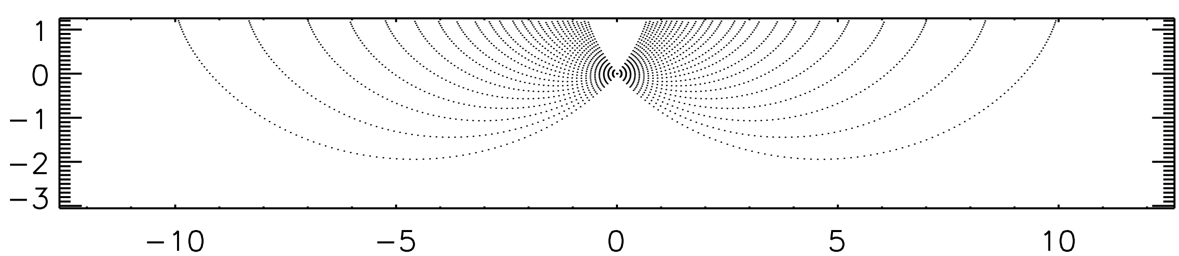

Figure 2: Focussing rays (launched from the focussing point).

2 Fast magnetosonic waves

In the presence of magnetic fields, one can use the dispersion relation for fast magnetosonic waves, $\omega = \boldsymbol{U}\cdot\boldsymbol{k} + \omega_{\rm ms}$. Here,

$$\omega_{\rm ms}^2 = \tfrac{1}{2}k^2(c_{\rm ms}^2 + c_{\rm m}^2) \tag{4}$$

with $c_{\rm ms}^2 = c_{\rm s}^2 + v_{\rm A}^2$, $v_{\rm A}^2 = \boldsymbol{v}_{\rm A}^2 = \boldsymbol{B}^2/\mu_0\rho$, and

$$c_{\rm m}^4 = c_{\rm ms}^4 - 4c_{\rm s}^2(\boldsymbol{v}_{\rm A}\cdot\boldsymbol{k})^2/k^2. \tag{5}$$

The group velocity is $\boldsymbol{v}_{\rm g} = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega$ and can be written as

$$\boldsymbol{v}_{\rm g} = \boldsymbol{U} + v_{\rm m}\hat{\boldsymbol{k}} + v_{\rm m}^{\|}\hat{\boldsymbol{k}}_{\|}, \tag{6}$$

with

$$v_{\rm m} = \frac{\omega_{\rm ms}}{k}\left(1 + c_{\rm n}^2\frac{k_{\|}^2}{\omega_{\rm ms}^2}\right), \quad v_{\rm m}^{\|} = -c_{\rm n}^2\frac{k_{\|}}{\omega_{\rm ms}}. \tag{7}$$

Note that

$$v_{\rm m} = \frac{\omega_{\rm ms}}{k}\left[1 + \frac{c_{\rm s}^2}{c_{\rm m}^2}\frac{(\boldsymbol{v}_{\rm A}\cdot\boldsymbol{k})^2}{\omega_{\rm ms}^2}\right], \tag{8}$$

and

$$v_{\rm m}^{\|}\hat{\boldsymbol{k}}_{\|} = -\frac{c_{\rm s}^2}{c_{\rm m}^2}\frac{v_{\rm A}k}{\omega_{\rm ms}}\boldsymbol{v}_{\rm A}. \tag{9}$$

$$c_{\rm m}^4 = (v_{\rm A}^2 - c_{\rm s}^2)^2 + 4v_{\rm A}^2 c_{\rm s}^2\, k_\perp^2/k^2 \tag{10}$$

so $\boldsymbol{v}_{\rm g} = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega = \boldsymbol{U} + \frac{1}{2}\omega_{\rm ms}^{-1}\boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega_{\rm ms}^2$ with[1]

$$\boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega_{\rm ms}^2 = (c_{\rm ms}^2 + c_{\rm m}^2)\,\boldsymbol{k} + 2c_{\rm n}^2(\boldsymbol{k}_\perp - \boldsymbol{k}k_\perp^2/k^2) \tag{13}$$

where

$$c_{\rm n}^2 = 2c_{\rm s}^2 v_{\rm A}^2/c_{\rm m}^2 \tag{14}$$

so

$$\boldsymbol{v}_{\rm g} = \boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega = \boldsymbol{U} + \left(\frac{\omega_{\rm ms}}{k^2} - \frac{c_{\rm n}^2}{\omega_{\rm ms}}\frac{k_\perp^2}{k^2}\right)\boldsymbol{k} + \frac{c_{\rm n}^2}{\omega_{\rm ms}}\boldsymbol{k}_\perp \tag{15}$$

or

$$\boldsymbol{v}_{\rm g} = \boldsymbol{U} + v_{\rm m}\hat{\boldsymbol{k}} + v_{\rm m}^{\perp}\hat{\boldsymbol{k}}_\perp, \tag{16}$$

with

$$v_{\rm m} = \frac{\omega_{\rm ms}}{k}\left(1 - c_{\rm n}^2\frac{k_\perp^2}{\omega_{\rm ms}^2}\right), \quad v_{\rm m}^{\perp} = c_{\rm n}^2\frac{k_\perp}{\omega_{\rm ms}}. \tag{17}$$

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[1]

$$\boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega_{\rm ms}^2 = \boldsymbol{k}\left\{c_{\rm ms}^2 + \left[(v_{\rm A}^2 - c_{\rm s}^2)^2 + 4v_{\rm A}^2 c_{\rm s}^2\frac{k_\perp^2}{k^2}\right]^{1/2}\right\} + \frac{\frac{1}{2}k^2}{2\left[(v_{\rm A}^2 - c_{\rm s}^2)^2 + 4v_{\rm A}^2 c_{\rm s}^2\frac{k_\perp^2}{k^2}\right]^{1/2}}\frac{8v_{\rm A}^2 c_{\rm s}^2}{k^4}(k^2\boldsymbol{k}_\perp - k_\perp^2\boldsymbol{k}) \tag{11}$$

or

$$\boldsymbol{\nabla}_{\boldsymbol{k}}\,\omega_{\rm ms}^2 = \boldsymbol{k}\left\{c_{\rm ms}^2 + \left[(v_{\rm A}^2 - c_{\rm s}^2)^2 + 4v_{\rm A}^2 c_{\rm s}^2\frac{k_\perp^2}{k^2}\right]^{1/2}\right\} + \frac{2v_{\rm A}^2 c_{\rm s}^2(\boldsymbol{k}_\perp - \boldsymbol{k}k_\perp^2/k^2)}{\left[(v_{\rm A}^2 - c_{\rm s}^2)^2 + 4v_{\rm A}^2 c_{\rm s}^2\frac{k_\perp^2}{k^2}\right]^{1/2}} \tag{12}$$