Handout 13: The dynamo instability

1 Roberts flow dynamos

Roberts (1972) investigated four incompressible spatially periodic steady flows with regard to their dynamo properties. More precisely, the flows vary periodically in the x and y directions, but are independent of z. We may write the corresponding velocities $\boldsymbol{u}$ so that the components u_x and u_y have in all four cases the form

$$u_x = v_0 \sin k_0 x \cos k_0 y\,, \quad u_y = -v_0 \cos k_0 x \sin k_0 y\,, \tag{1}$$

while the components u_z are different and given by

$$u_z = w_0 \sin k_0 x \sin k_0 y \qquad \text{(flow I)}, \tag{2}$$

$$u_z = w_0 \cos k_0 x \cos k_0 y \qquad \text{(flow II)}, \tag{3}$$

$$u_z = \tfrac{1}{2} w_0 (\cos 2k_0 x + \cos 2k_0 y) \qquad \text{(flow III)}, \tag{4}$$

$$u_z = w_0 \sin k_0 x \qquad \text{(flow IV)}, \tag{5}$$

where v_0, w_0 and k_0 are constants. In all four cases, Roberts found conditions under which dynamo action is possible, that is, magnetic fields may grow. The resulting magnetic fields survive xy averaging and are therefore amenable to mean-field treatment!

Figure 1 shows an example for the magnetic field evolution for Roberts flow I.[1] During class, we shall also look at $J_{\rm rms}/B_{\rm rms}$, which gives an indication about the typical wavenumber of the field. This will be different for different values of the magnetic Reynolds number, $\mathrm{Re}_M = u_{\rm rms}/\eta k_{\rm f}$. We shall also look at the work done against the Lorentz force, $-\langle \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) \rangle$.

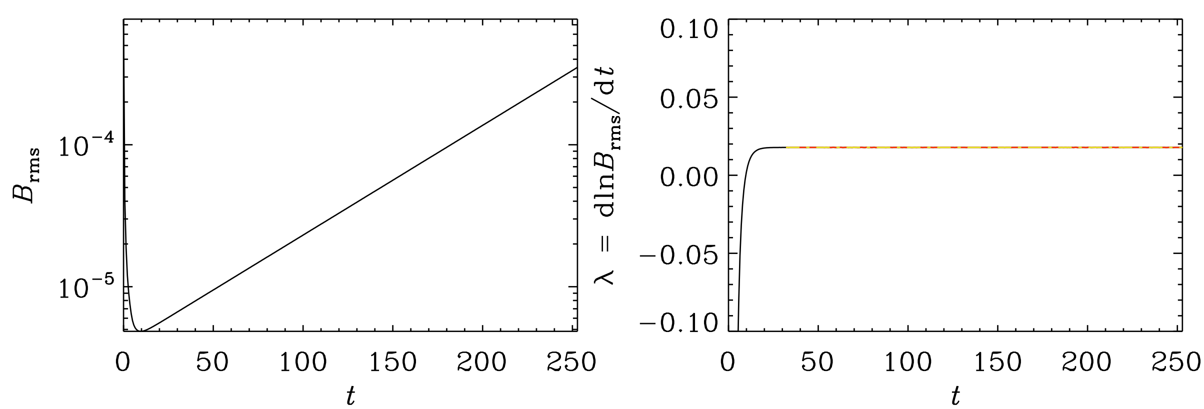

Figure 1: Growth of $B_{\rm rms}$ and the instantaneous growth rate.

2 Turbulent dynamos

"Playing" with kinematic flow dynamos has only limited usefulness. Real dynamos are nonlinear, so the magnetic field acts back on the flow and leads to saturation (and sometimes more complicated and unexpected phenomena).

An important diagnostics of turbulence in general are the kinetic and magnetic energy spectra. In incompressible (or nearly incompressible) isotropic turbulence one usually defines the spectral energy per

[1] `pencil-code/samples/kin-dynamo`

unit mass,

$$E(k,t) = \sum_{k_- < |\boldsymbol{k}| \leq k_+} |\hat{\boldsymbol{u}}(\boldsymbol{k},t)|^2, \tag{6}$$

where $k_\pm = k \pm \delta k/2$ mark a constant linear interval around wavenumber k, and the hat on $\boldsymbol{u}$ denotes the three-dimensional Fourier transformation in space. The spectral kinetic energy is normalized such that

$$\int_0^\infty E(k)\,\mathrm{d}k = \tfrac{1}{2}\langle \boldsymbol{u}^2 \rangle, \tag{7}$$

where angular brackets denote volume averaging. This equation shows that the dimension of $E(k,t)$ is $\mathrm{cm}^3\,\mathrm{s}^{-2}$, and $E(k)$ can be interpreted as the kinetic energy per unit mass and wavenumber.

Equivalent concepts and definitions also apply to the magnetic field $\boldsymbol{B}$, where one defines spectra of magnetic energy $M(k)$, magnetic helicity $H(k)$, and current helicity $C(k)$, which are normalized such that $\int M(k)\,\mathrm{d}k = \langle \boldsymbol{B}^2 \rangle / 2\mu_0$, where μ_0 is the vacuum permeability, $\int H(k)\,\mathrm{d}k = \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$, and $\int C(k)\,\mathrm{d}k = \langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle$. Here, $\boldsymbol{A}$ is the magnetic vector potential with $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}/\mu_0$ is the current density. The magnetic helicity and its spectrum are gauge-invariant because of the assumed periodicity of the underlying domain. In that case the addition of a gradient term, $\boldsymbol{\nabla}\Lambda$, in $\boldsymbol{A}$ has no effect, because $\langle \boldsymbol{\nabla}\Lambda \cdot \boldsymbol{B} \rangle = \langle \Lambda \boldsymbol{\nabla} \cdot \boldsymbol{B} \rangle = 0$, where we have used the condition that $\boldsymbol{B}$ is solenoidal.

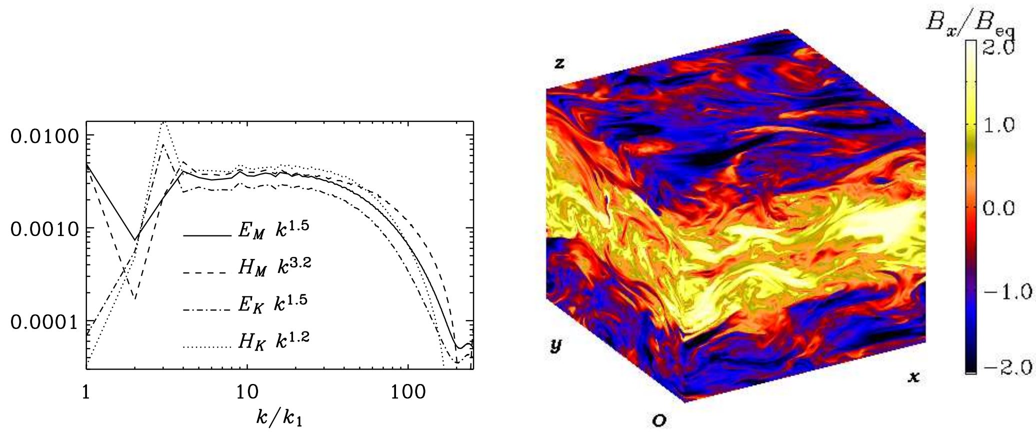

Figure 2: Left: Compensated time-averaged spectra of kinetic and magnetic energy, as well as of kinetic and magnetic helicity, for a run with $\mathrm{Re}_M = 600$. Right: Visualization of B_x on the periphery of the computational domain for a run with $\mathrm{Re}_M = 600$ and a resolution of 512^3 mesh points. Note the clear anisotropy with structures elongated in the direction of the field.

References

Roberts, G. O., "Dynamo action of fluid motions with two-dimensional periodicity," *Phil. Trans. Roy. Soc.* **A 271**, 411-454 (1972).

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