Handout 13b: The dynamo instability (cont'd)

In the wake of Cowling's antidynamo theorem[1] the Herzenberg dynamo played an important role as an early example of a dynamo where the existence of excited solutions could be proven rigorously. The Herzenberg dynamo does not attempt to model an astrophysical dynamo. Instead, it was complementary to some of the less mathematical and more phenomenological models at the time, such as Parker's migratory dynamo as well as the observational model of Babcock, and the semi-observational model of Leighton, all of which were specifically designed to describe the solar cycle.

1 Fast dynamos: the stretch-twist-fold picture

An elegant heuristic dynamo model illustrating the possibility of fast dynamos is what is often referred to as the Zeldovich 'stretch-twist-fold' (STF) dynamo (see Figure 1). We briefly outline it here, as it illustrates nicely several features of more realistic dynamos.

The dynamo algorithm starts with first stretching a closed flux rope to twice its length preserving its volume, as in an incompressible flow (A→B in Figure 1). The rope cross-section then decreases by factor two, and because of flux freezing the magnetic field doubles. In the next step, the rope is twisted into a figure eight (B→C in Figure 1) and then folded (C→D in Figure 1) so that now there are two loops, whose fields now point in the same direction and together occupy a similar volume as the original flux loop. The flux through this volume has now doubled. The last important step consists of merging the two loops into one (D→A in Figure 1), through small diffusive effects. This is important in order that the new arrangement cannot easily undo itself and the whole process becomes irreversible. The newly merged loops now become topologically the same as the original single loop, but now with the field strength scaled up by factor 2.

Repeating the algorithm n times, leads to the field in the flux loop growing by factor 2^n, or at a growth rate $\sim \ln 2/T$ where T is the time for the STF steps. This makes the dynamo potentially a fast dynamo, whose growth rate does not decrease with decreasing resistivity. Also note that the flux through a fixed 'Eulerian surface' grows exponentially, although the flux through any Lagrangian surface is nearly frozen; as it should be for small diffusivities.

The STF picture illustrates several other features: first we see that shear is needed to amplify the field at step A→B. However, without the twist part of the cycle, the field in the folded loop would cancel rather than add coherently. To twist the loop the motions need to leave the plane and go into the third dimension; this also means that field components perpendicular to the loop are generated, albeit being strong only temporarily during the twist part of the cycle. The source for the magnetic energy is the kinetic energy involved in the STF motions.

Most discussions of the STF dynamo assume implicitly that the last step of merging the twisted loops can be done at any time, and that the dynamo growth rate is not limited by this last step. This may well be true when the fields in the flux rope are not strong enough to affect the motions, that is, in the kinematic regime. However as the field becomes stronger, and if the merging process is slow, the Lorentz forces due to the small scale kinks and twists will gain in importance compared with the external forces associated with the driving of the loop as a whole. This may then limit the efficiency of the dynamo.

In this context one more feature deserves mentioning: if in the STF cycle one twists clockwise and folds, or twists counter-clockwise and folds one will still increase the field in the flux rope coherently. However, one would introduce opposite sense of writhe in these two cases, and so opposite internal twists. So, although the twist part of the cycle is important for the mechanism discussed here, the sense of twist can be random and does not require net helicity. This is analogous to a case when there is really only

[1] Larmor proposed in 1919 that the solar field might be generated by a self-excited dynamo. However, in 1933 Cowling published his antidynamo theorem, which states that two-dimensional (axisymmetric) magnetic fields cannot be sustained by dynamo action. Larmor (1934) responds to Cowling (1933) with the words "The view that I advanced briefly and tentatively long ago, which has come to be referred to as, perhaps too precisely, the self-exciting dynamo analogy, is still, so far as I know, the only foundation on which a gaseous body such as the Sun could possess a magnetic field: so that if it is demolished there could be no explanation of the Sun's magnetism even remotely in sight."
Cowling, T. G., "The magnetic field of sunspots," *Month. Not. Roy. Astron. Soc.* **94**, 39-48 (1933).
Larmor, J., "The magnetic field of sunspots," *Month. Not. Roy. Astron. Soc.* **94**, 469-471 (1934).

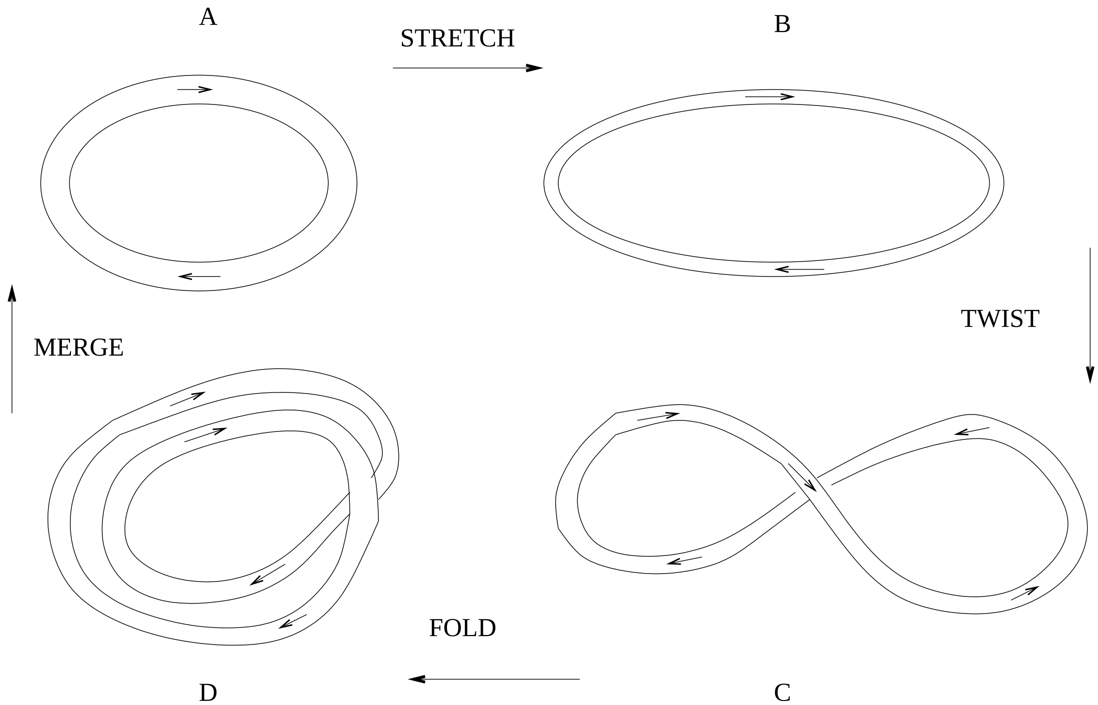

Figure 1: A schematic illustration of the stretch-twist-fold-merge dynamo.

a small scale dynamo, but one that requires finite kinetic helicity density locally. We should point out, however, that numerical simulations have shown that dynamos work and are potentially independent of magnetic Reynolds number even if the flow has zero kinetic helicity density everywhere.

If the twisted loops can be made to merge efficiently, the saturation of the STF dynamo would probably proceed differently. For example, the field in the loop may become too strong to be stretched and twisted, due to magnetic curvature forces. Another interesting way of saturation is that the incompressibility assumed for the motions may break down; as one stretches the flux loop the field pressure resists the decrease in the loop cross-section, and so the fluid density in the loop tends to decrease as one attempts to make the loop longer. (Note that it is B/ρ which has to increase during stretching.) The STF picture has inspired considerable work on various mathematical features of fast dynamos and some of this work can be found in the book by Childress and Gilbert which in fact has STF in its title!

2 Fast ABC-flow dynamos

ABC flows are solenoidal and fully helical with a velocity field given by

$$\boldsymbol{U} = \begin{pmatrix} C\sin kz + B\cos ky \\ A\sin kx + C\cos kz \\ B\sin ky + A\cos kx \end{pmatrix}. \tag{1}$$

When A, B, and C are all different from zero, the flow is no longer integrable and has chaotic streamlines. There is numerical evidence that such flows act as fast dynamos. The magnetic field has very small net magnetic helicity. This is a general property of any dynamo in the kinematic regime and follows from magnetic helicity conservation. Even in a nonlinear formulation of the ABC flow dynamo problem, where the flow is driven by a forcing function similar to Equation (1) the net magnetic helicity remains unimportant. This is however not surprising, because the development of net magnetic helicity requires sufficient scale separation, i.e. the wavenumber of the flow must be large compared with the smallest wavenumber in the box ($k = k_1$). If this is not the case, helical MHD turbulence behaves similarly to nonhelical turbulence. A significant scale separation also weakens the symmetries associated with the flow and the field, and leads to a larger kinematic growth rate, more compatible with the turnover time scale.

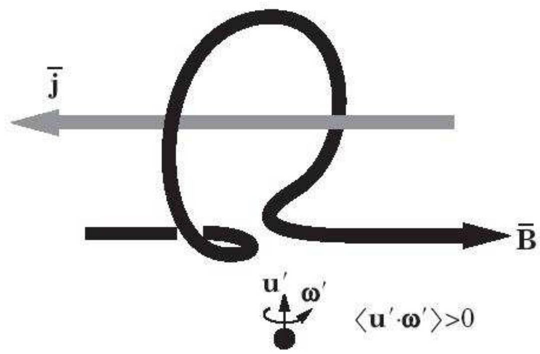

Figure 2: Production of positive writhe helicity by an uprising and expanding blob tilted in the clockwise direction by the Coriolis force in the southern hemisphere, producing a field-aligned current $\overline{\boldsymbol{J}}$ in the opposite direction to $\overline{\boldsymbol{B}}$.

Most of the recent work on nonlinear ABC flow dynamos has focused on the case with small scale separation and, in particular, on the initial growth and possible saturation mechanisms. In the kinematic regime, these authors find a near balance between Lorentz work and Joule dissipation. The balance originates primarily from small volumes where the strong magnetic flux structures are concentrated. The net growth of the magnetic energy comes about through stretching and folding of relatively weak field which occupies most of the volume. The mechanism for saturation could involve achieving a local pressure balance in these strong field regions.

3 Mean-field electrodynamics

In 1955 Parker first proposed the idea that the generation of a poloidal field, arising from the systematic effects of the Coriolis force (Figure 2), could be described by a corresponding term in the induction equation,

$$\frac{\partial \overline{\boldsymbol{B}}_{\rm pol}}{\partial t} = \boldsymbol{\nabla} \times \left(\alpha \overline{\boldsymbol{B}}_{\rm tor} + ... \right). \tag{2}$$

It is clear that such an equation can only be valid for averaged fields (denoted by overbars), because for the actual fields, the induced electromotive force (EMF) $\boldsymbol{U} \times \boldsymbol{B}$, would never have a component in the direction of $\boldsymbol{B}$. While being physically plausible, this approach only received general recognition and acceptance after Roberts and Stix (1972) translated the work of Steenbeck, Krause, Rädler (1966) into English. In those papers the theory for the α effect, as they called it, was developed and put on a mathematically rigorous basis. Furthermore, the α effect was also applied to spherical models of the solar cycle (with radial and latitudinal shear) and the geodynamo (with uniform rotation).

In mean field theory one solves the Reynolds averaged equations, using either ensemble averages, toroidal averages or, in cases in Cartesian geometry with periodic boundary conditions, two-dimensional (e.g. horizontal) averages. We thus consider the decomposition

$$\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}, \quad \boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}. \tag{3}$$

Here $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$ are the mean velocity and magnetic fields, while $\boldsymbol{u}$ and $\boldsymbol{b}$ are their fluctuating parts. These averages satisfy the Reynolds rules,

$$\overline{\boldsymbol{U}_1 + \boldsymbol{U}_2} = \overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{U}}_2, \quad \overline{\overline{\boldsymbol{U}}} = \overline{\boldsymbol{U}}, \quad \overline{\overline{\boldsymbol{U}}\boldsymbol{u}} = 0, \quad \overline{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}} = \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}, \tag{4}$$

$$\overline{\partial \boldsymbol{U}/\partial t} = \partial \overline{\boldsymbol{U}}/\partial t, \quad \overline{\partial \boldsymbol{U}/\partial x_i} = \partial \overline{\boldsymbol{U}}/\partial x_i. \tag{5}$$

Some of these properties are not shared by several other averages; for gaussian filtering $\overline{\overline{\boldsymbol{U}}} \neq \overline{\boldsymbol{U}}$, and for spectral filtering $\overline{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}} \neq \overline{\boldsymbol{U}}\,\overline{\boldsymbol{U}}$, for example. Note that $\overline{\overline{\boldsymbol{U}}} = \overline{\boldsymbol{U}}$ implies that $\overline{\boldsymbol{u}} = 0$.

In the remainder we assume that the Reynolds rules do apply. Averaging Equation (11) yields then the mean field induction equation,

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\mathcal{E}} - \eta \overline{\boldsymbol{J}}\right), \tag{6}$$

where

$$\overline{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}} \tag{7}$$

is the mean EMF. Finding an expression for the correlator $\overline{\mathcal{E}}$ in terms of the mean fields is a standard closure problem which is at the heart of mean field theory. In the two-scale approach one assumes that $\overline{\mathcal{E}}$ can be expanded in powers of the gradients of the mean magnetic field. This suggests the rather general expression

$$\mathcal{E}_i = \alpha_{ij}(\boldsymbol{g}, \hat{\boldsymbol{\Omega}}, \overline{\boldsymbol{B}}, ...)\overline{B}_j + \eta_{ijk}(\boldsymbol{g}, \hat{\boldsymbol{\Omega}}, \overline{\boldsymbol{B}}, ...)\partial \overline{B}_j / \partial x_k, \tag{8}$$

where the tensor components α_{ij} and η_{ijk} are referred to as turbulent transport coefficient. They depend on the stratification, angular velocity, and mean magnetic field strength. The dots indicate that the transport coefficients may also depend on correlators involving the small scale magnetic field, for example the current helicity of the small scale field. We have also kept only the lowest large scale derivative of the mean field; higher derivative terms are expected to be smaller.

The general form of the expression for $\overline{\mathcal{E}}$ can be determined by rather general considerations. For example, $\overline{\mathcal{E}}$ is a polar vector and $\overline{\boldsymbol{B}}$ is an axial vector, so α_{ij} must be a pseudo-tensor. The simplest pseudo-tensor of rank two that can be constructed using the unit vectors $\boldsymbol{g}$ (symbolic for radial density or turbulent velocity gradients) and $\hat{\boldsymbol{\Omega}}$ (angular velocity) is

$$\alpha_{ij} = \alpha_1 \delta_{ij}\, \boldsymbol{g} \cdot \hat{\boldsymbol{\Omega}} + \alpha_2 \hat{g}_i \hat{\Omega}_j + \alpha_3 \hat{g}_j \hat{\Omega}_i. \tag{9}$$

Note that the term $\boldsymbol{g} \cdot \hat{\boldsymbol{\Omega}} = \cos\theta$ leads to the co-sinusoidal dependence of α on latitude, θ, and a change of sign at the equator. Additional terms that are nonlinear in $\boldsymbol{g}$ or $\hat{\boldsymbol{\Omega}}$ enter if the stratification is strong or if the body is rotating rapidly. Likewise, terms involving $\overline{\boldsymbol{U}}$, $\overline{\boldsymbol{B}}$ and $\boldsymbol{b}$ may appear if the turbulence becomes affected by strong flows or magnetic fields. In the following section we discuss various approaches to determining the turbulent transport coefficients.

One of the most important outcomes of this theory is a quantitative formula for the coefficient α_1 in Equation (9) by Krause (1967)

$$\alpha_1\, \boldsymbol{g} \cdot \hat{\boldsymbol{\Omega}} = -\tfrac{16}{15} \tau_{\rm cor}^2 u_{\rm rms}^2 \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \ln(\rho u_{\rm rms}), \tag{10}$$

where $\tau_{\rm cor}$ is the correlation time, $u_{\rm rms}$ the root mean square velocity of the turbulence, and $\boldsymbol{\Omega}$ the angular velocity vector. The other coefficients are given by $\alpha_2 = \alpha_3 = -\alpha_1/4$. Throughout most of the solar convection zone, the product $\rho u_{\rm rms}$ decreases outward.[2] Therefore, $\alpha > 0$ throughout most of the northern hemisphere. In the southern hemisphere we have $\alpha < 0$, and α varies with colatitude θ like $\cos\theta$. However, this formula also predicts that α reverses sign very near the bottom of the convection zone where $u_{\rm rms} \to 0$. This is caused by the relatively sharp drop of $u_{\rm rms}$.

3.1 First order smoothing approximation

The first order smoothing approximation (FOSA) or, synonymously, the quasilinear approximation, or the second order correlation approximation is the simplest way of calculating turbulent transport coefficients. The approximation consists of linearizing the equations for the fluctuating quantities and ignoring quadratic terms that would lead to triple correlations in the expressions for the quadratic correlations. This technique has traditionally been applied to calculating the turbulent diffusion coefficient for a passive scalar or the turbulent viscosity (eddy viscosity).

Suppose we consider the induction equation. The equation for the fluctuating field can be obtained by subtracting Equation (6) from the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{J}/\sigma). \tag{11}$$

[2] This can be explained as follows: in the bulk of the solar convection zone the convective flux is approximately constant, and mixing length predicts that it is approximately $\rho u_{\rm rms}^3$. This in turn follows from $F_{\rm conv} \sim \rho u_{\rm rms} c_p \delta T$ and $u_{\rm rms}^2/H_p \sim g\delta T/T$ together with the expression for the pressure scale height $H_p = (1 - \frac{1}{\gamma}) c_p T/g$. Thus, since $\rho u_{\rm rms}^3 \approx$ const, we have $u_{\rm rms} \sim \rho^{-1/3}$ and $\rho u_{\rm rms}^3 \sim \rho^{2/3}$.

we obtain

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + \boldsymbol{u} \times \boldsymbol{b} - \overline{\mathcal{E}} - \eta \boldsymbol{j}\right), \tag{12}$$

where $\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{b} \equiv \boldsymbol{J} - \overline{\boldsymbol{J}}$ is the fluctuating current density. The first order smoothing approximation consists of *neglecting* the term $\boldsymbol{u} \times \boldsymbol{b}$ on the RHS of Equation (12), because it is nonlinear in the fluctuations. This can only be done if the fluctuations are small, which is a good approximation only under rather restrictive circumstances, for example if $R_{\rm m}$ is small. The term $\overline{\mathcal{E}}$ is also nonlinear in the fluctuations, but it is not a fluctuating quantity and gives therefore no contribution, and the $\overline{\boldsymbol{U}} \times \boldsymbol{b}$ is often neglected because of simplicity.

The neglect of the $\overline{\boldsymbol{U}}$ term may not be justified for systems with strong shear (e.g. for accretion discs) where the inclusion of $\overline{\boldsymbol{U}}$ itself could lead to a new dynamo effect, namely the shear–current effect. In the case of small $R_{\rm m}$, one can neglect both the $\boldsymbol{G}$ term and the time derivative of $\boldsymbol{b}$, resulting in a linear equation

$$\eta \boldsymbol{\nabla}^2 \boldsymbol{b} = -\boldsymbol{\nabla} \times \left(\boldsymbol{u} \times \overline{\boldsymbol{B}}\right). \tag{13}$$

This can be solved for $\boldsymbol{b}$, if $\boldsymbol{u}$ is given. $\overline{\mathcal{E}}$ can then be computed relatively easily.

However, in most astrophysical applications, $R_{\rm m} \gg 1$. In such a situation, FOSA is thought to still be applicable if the correlation time $\tau_{\rm cor}$ of the turbulence is small, such that $\tau_{\rm cor} u_{\rm rms} k_{\rm f} \ll 1$, where $u_{\rm rms}$ and $k_{\rm f}$ are typical velocity and correlation wavenumber, associated with the random velocity field $\boldsymbol{u}$. Under this condition, the ratio of the nonlinear term to the time derivative of $\boldsymbol{b}$ is argued to be $\sim (u_{\rm rms} k_{\rm f} b)/(b/\tau_{\rm cor}) = \tau_{\rm cor} u_{\rm rms} k_{\rm f} \ll 1$, and so $\boldsymbol{G}$ can be neglected (but see below). We then get

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{u} \times \overline{\boldsymbol{B}}\right). \tag{14}$$

To calculate $\overline{\mathcal{E}}$, we integrate $\partial \boldsymbol{b}/\partial t$ to get $\boldsymbol{b}$, take the cross product with $\boldsymbol{u}$, and average, i.e.

$$\overline{\mathcal{E}} = \overline{\boldsymbol{u}(t) \times \int_0^t \boldsymbol{\nabla} \times \left[\boldsymbol{u}(t') \times \overline{\boldsymbol{B}}(t')\right] \, {\rm d}t'}. \tag{15}$$

For clarity, we have suppressed the common $\boldsymbol{x}$ dependence of all variables. Using index notation, we have[3]

$$\overline{\mathcal{E}}_i(t) = \int_0^t \left[\hat{\alpha}_{ip}(t,t') \overline{B}_p(t') + \hat{\eta}_{ilp}(t,t') \overline{B}_{p,l}(t')\right] \, {\rm d}t', \tag{16}$$

with $\hat{\alpha}_{ip}(t,t') = \epsilon_{ijk} \overline{u_j(t) u_{k,p}(t')}$ and $\hat{\eta}_{ilp}(t,t') = -\epsilon_{ijp} \overline{u_j(t) u_l(t')}$, where we have used $\overline{B}_{l,l} = 0 = u_{l,l}$, and an additional term $\epsilon_{ijk} \overline{u_j(t) u_k(t')} \delta_{lp}$ in $\hat{\eta}_{ilp}(t,t')$ has been omitted, because it will soon drop out. In the statistically steady state, we can assume that $\hat{\alpha}_{ip}$ and $\hat{\eta}_{ilp}$ depend only on the time difference, $t - t'$. Assuming isotropy (again only for simplicity), these tensors must be proportional to the isotropic tensors δ_{ip} and ϵ_{ilp}, respectively, so we have

$$\overline{\mathcal{E}}(t) = \int_0^t \left[\hat{\alpha}(t-t') \overline{\boldsymbol{B}}(t') - \hat{\eta}_{\rm t}(t-t') \overline{\boldsymbol{J}}(t')\right] \, {\rm d}t', \tag{17}$$

where $\hat{\alpha}(t-t') = -\frac{1}{3} \overline{\boldsymbol{u}(t) \cdot \boldsymbol{\omega}(t')}$ and $\hat{\eta}_{\rm t}(t-t') = \frac{1}{3} \overline{\boldsymbol{u}(t) \cdot \boldsymbol{u}(t')}$ are integral kernels, and $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is the vorticity of the velocity fluctuation (see the footnote[4] for details).

If we assume the integral kernels to be proportional to the delta function, $\delta(t-t')$, or, equivalently, if $\overline{\boldsymbol{B}}$ can be considered a slowly varying function of time, one arrives at

$$\overline{\mathcal{E}} = \alpha \overline{\boldsymbol{B}} - \eta_{\rm t} \overline{\boldsymbol{J}} \tag{18}$$

with

$$\alpha = -\tfrac{1}{3} \int_0^t \overline{\boldsymbol{u}(t) \cdot \boldsymbol{\omega}(t')} \, {\rm d}t' \approx -\tfrac{1}{3} \tau_{\rm cor} \overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}, \tag{19}$$

[3] Note that $[\boldsymbol{u} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})]_i = \epsilon_{ijk} \epsilon_{klm} \epsilon_{mnp} u_j \partial_l (u_n \overline{B}_p) = \hat{\alpha}_{ip} \overline{B}_p + \hat{\eta}_{ilp} \overline{B}_{p,l}$, where commas denote partial differentiation and time arguments in $\hat{\alpha}_{ip} = \hat{\alpha}_{ip}(t,t')$, $\hat{\eta}_{ilp} = \hat{\eta}_{ilp}(t,t')$, and $\overline{B}_p = \overline{B}_p(t')$ has been omitted. Contracting $\epsilon_{klm} \epsilon_{mnp} = \delta_{kn} \delta_{lp} - \delta_{kp} \delta_{ln}$ gives $\hat{\alpha}_{ip} = \epsilon_{ijk} (\overline{u_j u_{k,p}} - \overline{u_j u_{l,l}} \delta_{kp}$, and $\hat{\eta}_{ilp} = \epsilon_{ijk} (\overline{u_j u_k} \delta_{lp} - \overline{u_j u_l} \delta_{kp})$.

[4] Note that under isotropy we have $\hat{\alpha}_{ip} = \hat{\alpha} \delta_{ip}$ and $\hat{\eta}_{ilp} = \hat{\eta}_{\rm t} \epsilon_{ilp}$. Multiplying these equations by δ_{ip} and ϵ_{ilp}, respectively, and noting that $\delta_{ip} \delta_{ip} = 3$ and $\epsilon_{ilp} \epsilon_{ilp} = 6$ we have $\hat{\alpha} = \frac{1}{3} \hat{\alpha}_{ip} \delta_{ip} = \frac{1}{3} \overline{u_j \epsilon_{jki} u_{k,i}} = -\frac{1}{3} \overline{\boldsymbol{u} \cdot \boldsymbol{\omega}}$, and $\hat{\eta}_{\rm t} = \frac{1}{6} \hat{\eta}_{ilp} \epsilon_{ilp} = -\frac{1}{6} \epsilon_{ijp} \overline{u_j u_l} \epsilon_{ilp} = -\frac{1}{3} \overline{\boldsymbol{u}^2}$, where we have used $\epsilon_{ijp} \epsilon_{ilp} = 2 \delta_{jl}$, and the t and t' arguments have been omitted.

$$\eta_{\rm t} = \tfrac{1}{3} \int_0^t \overline{\boldsymbol{u}(t) \cdot \boldsymbol{u}(t')} \, {\rm d}t' \approx \tfrac{1}{3} \tau_{\rm cor} \overline{\boldsymbol{u}^2}. \tag{20}$$

When t becomes large, the main contribution to these two expressions comes only from late times, $t - t' \ll t$, because then the contributions from early times are no longer strongly correlated with $\boldsymbol{u}(t)$. By using FOSA we have thus solved the problem of expressing $\overline{\mathcal{E}}$ in terms of the mean field. The turbulent transport coefficients α and $\eta_{\rm t}$ depend, respectively, on the helicity and the energy density of the turbulence.

3.2 MTA – the 'minimal' τ approximation

The 'minimal' τ approximation is a simplified version of the τ approximation as it has been introduced by Orszag (1970) and used by Pouquet, Frisch and Léorat (1976) in the context of the Eddy Damped Quasi Normal Markovian (EDQNM) approximation. In that case a damping term is introduced in order to express fourth order moments in terms of third order moments. In the τ approximation, as introduced by Vainshtein and Kitchatinov and Kleeorin and Rogachevskii one approximates triple moments in terms of quadratic moments via a wavenumber-dependent relaxation time $\tau(k)$. The 'minimal' τ approximation (MTA), as it is introduced by Blackman and Field is applied in real space in the two-scale approximation. We will refer to both the above types of closures (where triple moments are approximated in terms of quadratic moments and a relaxation time τ) as the 'minimal' τ approximation or MTA.

There are some technical similarities between FOSA and the minimal τ approximation. The main advantage of the τ approximation is that the fluctuations do *not* need to be small and so the triple correlations are no longer neglected. Instead, it is assumed (and this can be and has been tested using simulations) that the one-point triple correlations are proportional to the quadratic correlations, and that the proportionality coefficient is an inverse relaxation time that can in principle be scale (or wavenumber) dependent.

In this approach, one begins by considering the time derivative of $\overline{\mathcal{E}}$,

$$\frac{\partial \overline{\mathcal{E}}}{\partial t} = \overline{\boldsymbol{u} \times \dot{\boldsymbol{b}}} + \overline{\dot{\boldsymbol{u}} \times \boldsymbol{b}}, \tag{21}$$

where a dot denotes a time derivative. For $\dot{\boldsymbol{b}}$, we substitute Equation (12) and for $\dot{\boldsymbol{u}}$, we use the Euler equation for the fluctuating velocity field,

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0} \boldsymbol{\nabla} p + \boldsymbol{f} + \boldsymbol{F}_{\rm vis} + \boldsymbol{H}, \tag{22}$$

where $\boldsymbol{H} = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \overline{\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}}$ is the nonlinear term, $\boldsymbol{f}$ is a stochastic forcing term (with zero divergence), and $\boldsymbol{F}_{\rm vis}$ is the viscous force. We have also assumed for the present that there is no mean flow ($\overline{\boldsymbol{U}} = 0$), and have considered the kinematic regime where the Lorentz force is set to zero. All these restrictions can in principle be lifted (see below). For an incompressible flow, the pressure term can be eliminated in the standard fashion in terms of the projection operator.

Now since $\boldsymbol{f}$ does not correlate with $\boldsymbol{b}$, the only contribution to $\overline{\dot{\boldsymbol{u}} \times \boldsymbol{b}}$, is the small viscous term and the triple correlation involving $\boldsymbol{b}$ and $\boldsymbol{H}$. The $\overline{\boldsymbol{u} \times \dot{\boldsymbol{b}}}$ term however has non-trivial contributions. We get

$$\frac{\partial \overline{\mathcal{E}}}{\partial t} = \tilde{\alpha} \, \overline{\boldsymbol{B}} - \tilde{\eta}_{\rm t} \, \overline{\boldsymbol{J}} - \frac{\overline{\mathcal{E}}}{\tau}, \tag{23}$$

where the last term subsumes the effects of all triple correlations, and

$$\tilde{\alpha} = -\tfrac{1}{3} \overline{\boldsymbol{u} \cdot \boldsymbol{\omega}} \quad \text{and} \quad \tilde{\eta}_{\rm t} = \tfrac{1}{3} \overline{\boldsymbol{u}^2} \quad \text{(kinematic theory)} \tag{24}$$

are coefficients that are closely related to the usual α and $\eta_{\rm t}$ coefficients in Equation (18). We recall that in this *kinematic* calculation the Lorentz force, and in fact the entire $\dot{\boldsymbol{u}}$ equation in Equation (21) has been ignored. Its inclusion turns out to be extremely important: it leads to the emergence of a small scale magnetic correction term in the expression for $\tilde{\alpha}$.

One normally neglects the explicit time derivative of $\overline{\mathcal{E}}$, and arrives then at almost the same expression as Equation (18), except that now one deals directly with one-point correlation functions and not only via an approximation. Furthermore, the explicit time derivative can in principle be kept, although it

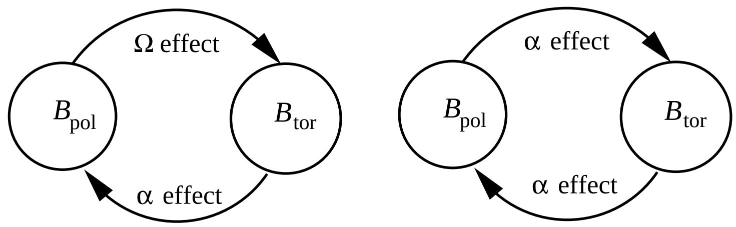

Figure 3: Mutual regeneration of poloidal and toroidal fields in the case of the $\alpha\Omega$ dynamo (left) and the α^2 dynamo (right).

becomes unimportant on time scales long compared with τ. In comparison with Equation (17), we note that if one assumes $\hat{\alpha}(t-t')$ and $\hat{\eta}_{\rm t}(t-t')$ to be proportional to $\exp[-(t-t')/\tau]$ for $t>t'$ (and zero otherwise), one recovers Equation (23) with the relaxation time τ playing now the role of a correlation time.

4 α^2 and $\alpha\Omega$ dynamos: simple solutions

For astrophysical purposes one is usually interested in solutions in spherical or oblate (disc-like) geometries. However, in order to make contact with turbulence simulations in a periodic box, solutions in simpler Cartesian geometry can be useful. Cartesian geometry is also useful for illustrative purposes. In this subsection we review some simple cases.

Mean field dynamos are traditionally divided into two groups; $\alpha\Omega$ and α^2 dynamos. The Ω effect refers to the amplification of the toroidal field by shear (i.e. *differential* rotation) and its importance for the sun was recognized very early on. Such shear also naturally occurs in disk galaxies, since they are differentially rotating systems. However, it is still necessary to regenerate the poloidal field. In both stars and galaxies the α effect is the prime candidate. This explains the name $\alpha\Omega$ dynamo; see the left hand panel of Figure 3. However, large scale magnetic fields can also be generated by the α effect alone, so now also the toroidal field has to be generated by the α effect, in which case one talks about an α^2 dynamo; see the right hand panel of Figure 3. (The term $\alpha^2\Omega$ model is discussed at the end of Section 4.2.)

4.1 α^2 dynamo in a periodic box

We assume that there is no mean flow, i.e. $\overline{\boldsymbol{U}}=0$, and that the turbulence is homogeneous, so that α and $\eta_{\rm t}$ are constant. The mean field induction equation then reads

$$\frac{\partial\overline{\boldsymbol{B}}}{\partial t}=\alpha\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}+\eta_{\rm T}\nabla^2\overline{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\overline{\boldsymbol{B}}=0, \tag{25}$$

where $\eta_{\rm T}=\eta+\eta_{\rm t}$ is the sum of microscopic and turbulent magnetic diffusivity. We can seek solutions of the form

$$\overline{\boldsymbol{B}}(\boldsymbol{x})=\mathrm{Re}\left[\hat{\boldsymbol{B}}(\boldsymbol{k})\,\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\lambda t)\right]. \tag{26}$$

This leads to the eigenvalue problem $\lambda\hat{\boldsymbol{B}}=\alpha\mathrm{i}\boldsymbol{k}\times\hat{\boldsymbol{B}}-\eta_{\rm T}k^2\hat{\boldsymbol{B}}$, which can be written in matrix form as

$$\lambda\hat{\boldsymbol{B}}=\begin{pmatrix}-\eta_{\rm T}k^2 & -\mathrm{i}\alpha k_z & \mathrm{i}\alpha k_y\\ \mathrm{i}\alpha k_z & -\eta_{\rm T}k^2 & -\mathrm{i}\alpha k_x\\ -\mathrm{i}\alpha k_y & \mathrm{i}\alpha k_x & -\eta_{\rm T}k^2\end{pmatrix}\hat{\boldsymbol{B}}. \tag{27}$$

This leads to the dispersion relation, $\lambda=\lambda(\boldsymbol{k})$, given by

$$(\lambda+\eta_{\rm T}k^2)\left[(\lambda+\eta_{\rm T}k^2)^2-\alpha^2k^2\right]=0, \tag{28}$$

with the three solutions

$$\lambda_0=-\eta_{\rm T}k^2,\quad \lambda_\pm=-\eta_{\rm T}k^2\pm|\alpha k|. \tag{29}$$

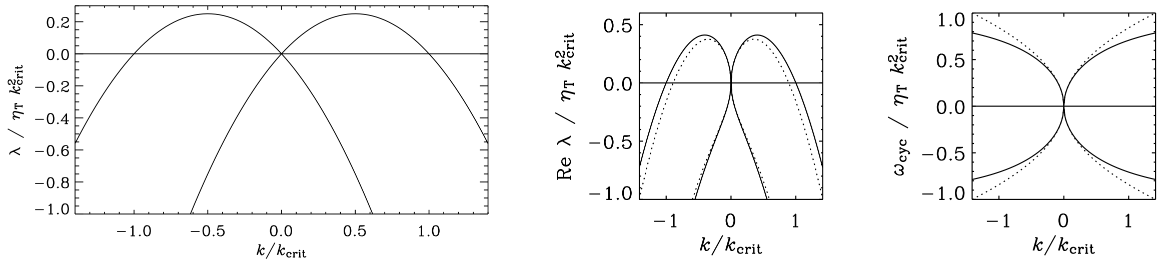

Figure 4: *Left panel:* Dispersion relation for α^2 dynamo, where $k_{\rm crit}=\alpha/\eta_{\rm T}$. *Middle and Right panels:* Dispersion relation for $\alpha^2\Omega$ dynamo with $\alpha k_{\rm crit}/S=0.35$. The dotted line gives the result for the $\alpha\Omega$ approximation Equations (34) and (35). The axes are normalized using $k_{\rm crit}$ for the full $\alpha^2\Omega$ dynamo equations.

The eigenfunction corresponding to the eigenvalue $\lambda_0=-\eta_{\rm T}k^2$ is proportional to $\boldsymbol{k}$, but this solution is incompatible with solenoidality and has to be dropped. The two remaining branches are shown in Figure 4.

Unstable solutions ($\lambda>0$) are possible for $0<\alpha k<\eta_{\rm T}k^2$. For $\alpha>0$ this corresponds to the range

$$0<k<\alpha/\eta_{\rm T}\equiv k_{\rm crit}. \tag{30}$$

For $\alpha<0$, unstable solutions are obtained for $k_{\rm crit}<k<0$. The maximum growth rate is at $k=\frac{1}{2}k_{\rm crit}$. Such solutions are of some interest, because they have been seen as an additional hump in the magnetic energy spectra from fully three-dimensional turbulence simulations.

4.2 $\alpha\Omega$ dynamo in a periodic box

Next we consider the case with linear shear, and assume $\overline{\boldsymbol{U}}=(0,Sx,0)$, where $S=$ const. This model can be applied as a local model to both accretion discs (x is radius, y is longitude, and z is the height above the midplane) and to stars (x is latitude, y is longitude, and z is radius). For Keplerian discs, the shear is $S=-\frac{3}{2}\Omega$, while for the sun (taking here only radial differential rotation into account) $S=r\partial\Omega/\partial r\approx+0.1\Omega_\odot$ near the equator.

For simplicity we consider axisymmetric solutions, i.e. $k_y=0$. The eigenvalue problem takes then the form

$$\lambda\hat{\boldsymbol{B}}=\begin{pmatrix}-\eta_{\rm T}k^2 & -{\rm i}\alpha k_z & 0\\ {\rm i}\alpha k_z+S & -\eta_{\rm T}k^2 & -{\rm i}\alpha k_x\\ 0 & {\rm i}\alpha k_x & -\eta_{\rm T}k^2\end{pmatrix}\hat{\boldsymbol{B}}, \tag{31}$$

where $\eta_{\rm T}=\eta+\eta_{\rm t}$ and $\boldsymbol{k}^2=k_x^2+k_z^2$. The dispersion relation is now

$$(\lambda+\eta_{\rm T}k^2)\left[(\lambda+\eta_{\rm T}k^2)^2+{\rm i}\alpha Sk_z-\alpha^2k^2\right]=0, \tag{32}$$

with the solutions

$$\lambda_\pm=-\eta_{\rm T}k^2\pm(\alpha^2k^2-{\rm i}\alpha Sk_z)^{1/2}. \tag{33}$$

Again, the eigenfunction corresponding to the eigenvalue $\lambda_0=-\eta_{\rm T}k^2$ is not compatible with solenoidality and has to be dropped. The two remaining branches are shown in the middle- and right-hand panel of Figure 4, together with the *approximate* solutions (valid for $\alpha k_z/S\ll 1$)

$$\mathrm{Re}\lambda_\pm\approx-\eta_{\rm T}k^2\pm|\tfrac{1}{2}\alpha Sk_z|^{1/2}, \tag{34}$$

$$\mathrm{Im}\lambda_\pm\equiv-\omega_{\rm cyc}\approx\pm|\tfrac{1}{2}\alpha Sk_z|^{1/2}, \tag{35}$$

where we have made use of the fact that ${\rm i}^{1/2}=(1+{\rm i})/\sqrt{2}$.

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