Handout 14: Nonlinear Water Waves

The KdV equation can be written in the form

$$\dot{u} + uu' + u''' = 0 \tag{1}$$

1 Energy conservation in KdV

Among the several other conservation laws, energy conservation is an important one. To contrast the effects of viscosity with dispersive effects, let us write

$$\dot{u} = -uu' + \nu u'' - \mu u''', \tag{2}$$

where we have introduced viscosity ν and "dispersivity" μ. To compute energy conservation, let us write

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right) \equiv u\dot{u} = -u^2u' + \nu uu'' - \mu uu'''. \tag{3}$$

The advection operator does not change the energy, because

$$\int u^2u' \,\mathrm{d}x = \int \left(\tfrac{1}{3}u^3\right)' \,\mathrm{d}x = 0. \tag{4}$$

But even the dispersive term does not change the energy:

$$\int uu''' \,\mathrm{d}x = \int \left(uu''\right)' \,\mathrm{d}x - \int \left(u'u''\right) \,\mathrm{d}x = \int \left(uu''\right)' \,\mathrm{d}x - \int \left(u'^2\right)' \,\mathrm{d}x = \int \left(uu'' - u'^2\right)' \,\mathrm{d}x = 0. \tag{5}$$

By comparison,

$$\int uu'' \,\mathrm{d}x = \int \left(uu'\right)' \,\mathrm{d}x - \int \left(u'^2\right) \,\mathrm{d}x = -\int \left(u'^2\right) \,\mathrm{d}x \neq 0, \tag{6}$$

does lead to energy dissipation.

2 Solution

To determine the solution, we make what is called an ansatz, namely

$$u = \frac{A}{\cosh^2[a(x-ct)]}, \tag{7}$$

which has 3 unknowns that can be determined such that Equation (1) is obeyed. We now compute every term in turn and begin with

$$\dot{u} = +2Aac\frac{\sinh[a(x-ct)]}{\cosh^3[a(x-ct)]}. \tag{8}$$

Next to compute uu' and later u''', we need

$$u' = -2Aa\frac{\sinh[a(x-ct)]}{\cosh^3[a(x-ct)]}. \tag{9}$$

We see that with each differentiation we pull out a factor a. To simplify notation let us now introduce

$$\theta = x(x-ct) \tag{10}$$

for the argument of the cosh and sinh functions, so

$$u'' = -2Aa^2\left(-3\,\frac{\sinh^2\theta}{\cosh^4\theta} + \frac{1}{\cosh^2\theta}\right). \tag{11}$$

Finally, we compute

$$u''' = -2Aa^3 \left[-3\left(-4\,\frac{\sinh^3\theta}{\cosh^5\theta} + 2\,\frac{\sinh\theta}{\cosh^3\theta}\right) - 2\,\frac{\sinh\theta}{\cosh^3\theta}\right], \tag{12}$$

which combines to

$$u''' = -2Aa^3 \left(12\,\frac{\sinh^3\theta}{\cosh^5\theta} - 8\,\frac{\sinh\theta}{\cosh^3\theta}\right). \tag{13}$$

Making use of the relation $\cosh^2 - \sinh^2 x = 1$, i.e., $\sinh^2 x = \cosh^2 - 1$, we have

$$u''' = -2Aa^3 \left(12\,\frac{\sinh\theta\,(\cosh^2\theta - 1)}{\cosh^5\theta} - 8\,\frac{\sinh\theta}{\cosh^3\theta}\right). \tag{14}$$

or

$$u''' = -2Aa^3 \left(12\,\frac{\sinh\theta}{\cosh^3\theta} - 12\,\frac{\sinh\theta}{\cosh^5\theta} - 8\,\frac{\sinh\theta}{\cosh^3\theta}\right). \tag{15}$$

and therefore

$$u''' = -2Aa^3 \left(4\,\frac{\sinh\theta}{\cosh^3\theta} - 12\,\frac{\sinh\theta}{\cosh^5\theta}\right). \tag{16}$$

Putting now everything together, we have

$$\dot{u} + uu' + u''' = 2aA\frac{\sinh\theta}{\cosh^3\theta}\left[(c - 4a^2) + (-A + 12a^2)\,\frac{1}{\cosh^2\theta}\right]. \tag{17}$$

The rhs can only vanish if

$$c = 4a^2 = A/3. \tag{18}$$

We also see that, if we were to introduce a parameter μ in front of the dispersive term, i.e.,

$$\dot{u} + uu' + \mu u''' = 0, \tag{19}$$

the solution would read

$$c = 4a^2/\mu = A/3. \tag{20}$$

so the relation $A = 3c$ is not altered, but just the width changes.

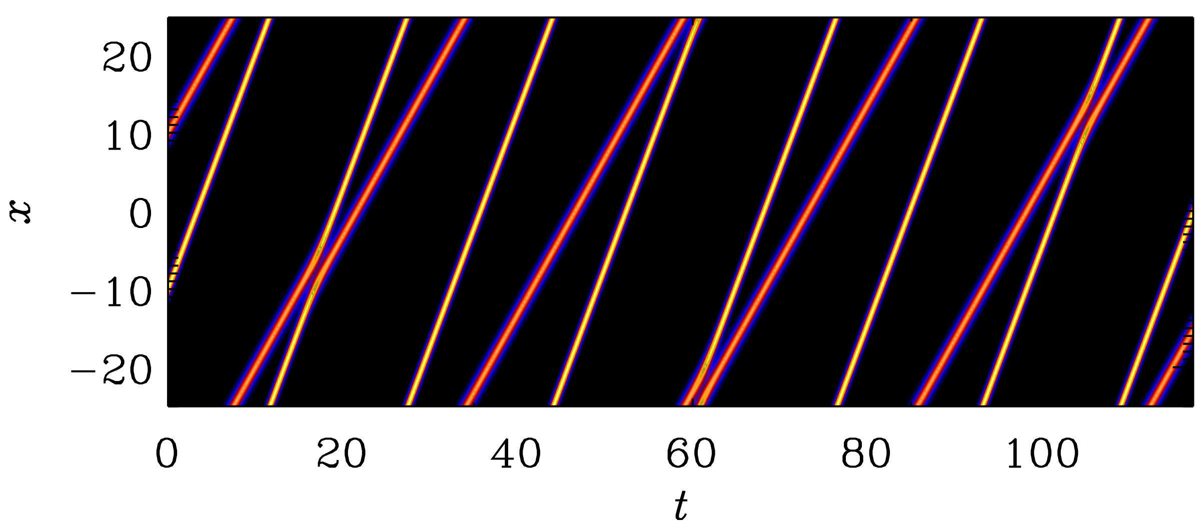

Figure 1: xt diagram for $c_1 = 3$ and $c_2 = 2$.

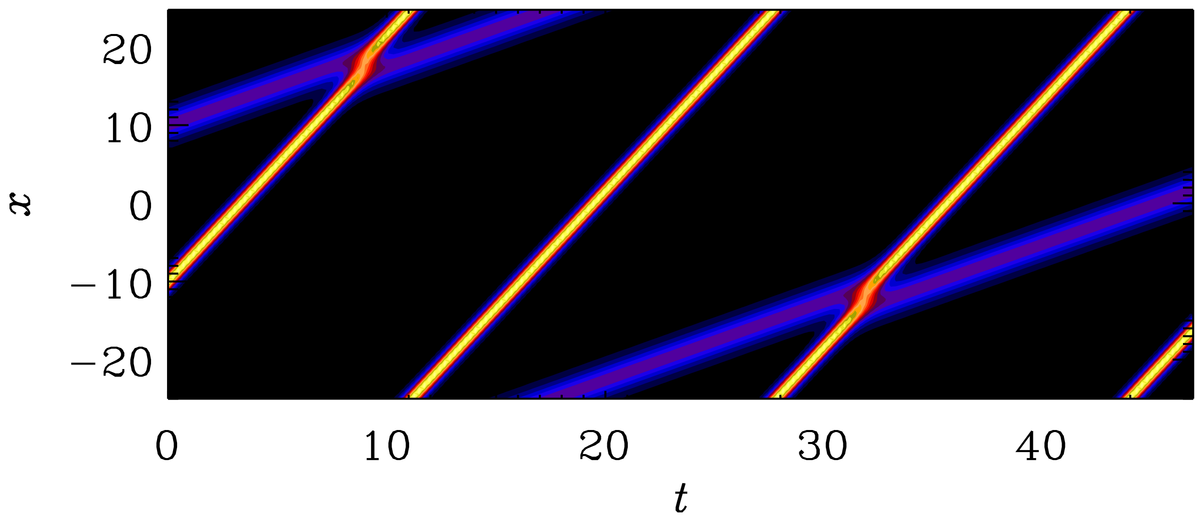

Figure 2: xt diagram for $c_1 = 3$ and $c_2 = 1$.

3 Numerical solutions

To compute numerical solutions of the KdV equation, one can just use a high-order finite difference scheme and represent first derivative on a discrete mesh as

$$f'_i = (-f_{i-3} + 9f_{i-2} - 45f_{i-1} + 45f_{i+1} - 9f_{i+2} + f_{i+3})/(60\delta x), \tag{21}$$

and the third derivative as

$$f'''_i = (+f_{i-3} - 8f_{i-2} + 13f_{i-1} - 13f_{i+1} + 8f_{i+2} - f_{i+3})/(8\delta x^3). \tag{22}$$

Both formulae have a stencil width of three in each direction, but the first derivative is sixth order and the third one is only second order accurate. A third derivative that is also sixth order has a stencil width of four:

$$f'''_i = (+7f_{i-4} - 72f_{i-3} + 338f_{i-2} - 488f_{i-1} + 488f_{i+1} - 338f_{i+2} + 72f_{i+3} - 7f_{i-4})/(240\delta x^3). \tag{23}$$

The equations are advanced in time by a time-stepping scheme. It is advantageous to choose a high-order scheme, e.g., a third order scheme. Higher order schemes also allow for a longer time step, which allows the code still to be stable. The maximum possible time step scales in a well-defined way with the parameters in the simulation. For pure advection, this is known as the Courant–Friedrichs–Lewy condition, i.e., $\delta t < C_{\rm CFL}\delta x/u_{\rm max}$. If viscosity is important, it can constrain the time step further, and on dimensional grounds it must be $\delta t < C_{\rm visc}\delta x^2/\nu$, and likewise for dispersion, $\delta t < C_{\rm disp}\delta x^3/\mu$. In practice, we can take the minimum of all three or more such constraints, i.e.,

$$\delta t_{\rm max} = \min\left[C_{\rm CFL}\delta x/u_{\rm max}, C_{\rm visc}\delta x^2/\nu, C_{\rm disp}\delta x^3/\mu\right]. \tag{24}$$

For the code at hand, we found empirically $C_{\rm CFL} \approx 0.9$, $C_{\rm visc} \approx 0.1$, and $C_{\rm disp} \approx 0.3$.

Solitons cannot be superimposed just like that. Exact two-soliton solutions do actually exist, and if they are fare enough apart initially, the addition of two solution is good enough. In Figures 1 and 2 we show examples of soliton collisions. One clearly sees that the actual interaction is not just the sum of two. Also, there is always a phase shift.

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