Handout 14b: Nonlinear Water Waves (cont'd)

The KdV equation can be derived under the restrictions of shallow water (wavelength longer ℓ long compared with depth h) and small (but finite) amplitude a compared with h, i.e.,

$$a \ll h \ll \ell. \tag{1}$$

One assumes an inviscid irrotational flow $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, so the governing equation is the Bernoulli equation,

$$\partial_t \phi + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + P/\rho + gz = 0, \qquad \nabla^2\phi = 0. \tag{2}$$

We first need to discuss boundary conditions.

1 Boundary conditions

At rest, the water surface is assumed to be at $z =$, so the bottom is at $z = -h$, and the normal velocity vanishes there, so

$$\phi_{,z} = 0 \qquad \text{(at } z = -h). \tag{3}$$

Next, the surface is assumed to be at $z = \zeta$. Since the pressure vanishes zero, we have

$$\partial_t \phi^{\rm s} + \tfrac{1}{2}(\boldsymbol{\nabla}\phi^{\rm s})^2 + g\zeta = 0. \tag{4}$$

where the superscript s refers to the surface. The location of the surface is described by the function $\zeta = \zeta(x, t)$, and we assume that

$${\rm D}\zeta/{\rm D}t = u_z \equiv \phi_{,z}. \tag{5}$$

Since ${\rm D}\zeta/{\rm D}t = \partial_t\zeta + u_x\zeta_{,x} = \partial_t\zeta + \phi_{,x}\zeta_{,x}$, we can also write

$$\partial_t\zeta + \phi_{,x}\zeta_{,x} = \phi_{,z} \qquad \text{(at the surface).} \tag{6}$$

We note in passing that the linearized form of the two equations can be combined to $\partial_t^2\phi + g\phi_{,z} = 0$, which is an equation we have encountered in handout 11, see Eq. (16) of that handout.

2 Linear wave solutions

Assuming wave-like solutions of the form

$$\phi = f(z)\sin(kx - \omega t), \tag{7}$$

which satisfy $\nabla^2\phi = 0$, the f has to be of the form $f = f_1 e^{kz} + f_2 e^{-kz}$. To obey $f_{,z} = 0$, we have to have $f_1 k e^{kz} - f_2 k e^{-kz} = 0$ at $z = -h$, and thus $f_1 k e^{-kh} - f_2 k e^{kh} = 0$, or $f_2/f_1 = e^{-2kh}$ and thus

$$f(z) = A\left(e^{kz} + e^{-kz-2kh}\right) = Ae^{-kh}\left(e^{k(z+h)} + e^{-k(z+h)}\right), \tag{8}$$

and therefore $f(z) = 2Ae^{-kh}\cosh k(z+h)$.
Inserting $\phi = 2Ae^{-kh}\cosh k(z+h)\sin(kx-\omega t)$ into $\partial_t^2\phi + g\phi_{,z} = 0$, and noting that $f_{,z} = 2Ake^{-kh}\sinh k(z+h)$, we have

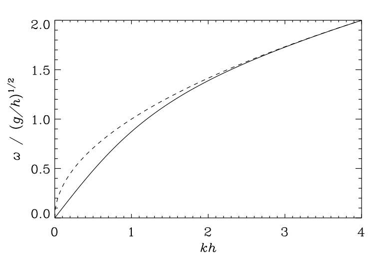

Figure 1: Dispersion relation. The dashed line shows the result without the tanh factor.

$$-\omega^2\, 2Ae^{-kh}\cosh k(z+h)\sin(kx-\omega t) + 2Agk\, e^{-kh}\cosh k(z+h)\sin(kx-\omega t) = 0, \tag{9}$$

and therefore $\omega^2 = gk\,\tanh k(z+h)$ at the surface at $z = 0$, so

$$\omega^2 = gk\,\tanh kh. \tag{10}$$

This is shown in Figure 1 and compared with $\omega^2 = gk$.

3 Perturbative nonlinear wave equation

We now make the following ansatz for ϕ, which must obey $\nabla^2\phi = 0$,

$$\phi = \sum_{n=0}^{\infty} z^n \phi_n(x,t). \tag{11}$$

To work with this, it is useful to write out the first few terms:

$$\phi = \phi_0 + z\phi_1 + z^2\phi_2 + z^3\phi_3 + z^4\phi_4 + ... \tag{12}$$

In order that this satisfies $\nabla^2\phi = 0$, let us write down the second x and z derivatives,

$$\phi_{,xx} = \phi_{0,xx} + z\phi_{1,xx} + z^2\phi_{2,xx} + z^3\phi_{3,xx} + z^4\phi_{4,xx} + ... \tag{13}$$

$$\phi_{,zz} = 2\phi_2 + 3\cdot 2\, z\phi_3 + 4\cdot 3\, z^2\phi_4 + ... \tag{14}$$

Matching equal powers of z leads to the following *recursive relations*

$$\phi_{0,xx} + 2\,\phi_2 = 0, \tag{15}$$

$$\phi_{1,xx} + 3\cdot 2\,\phi_3 = 0, \tag{16}$$

$$\phi_{2,xx} + 4\cdot 3\,\phi_4 = 0, \tag{17}$$

or, more generally

$$\phi_{n,xx} + (n+2)(n+1)\,\phi_{n+2} = 0. \tag{18}$$

Next, making use of the bottom boundary condition $u_z = 0$, i.e., $\phi_{,z} = 0$, we find $\phi_1 = 0$, and, because of Equation (18), all odd terms vanish, i.e., $\phi_3 = \phi_5 = ... = 0$. With this, we can now write ϕ as

$$\phi = \phi_0 + z^2\,\phi_2 + z^4\,\phi_4 + ... \tag{19}$$

Inserting the recursive relations, we have $\phi_2 = -\frac{1}{2}\phi_{0,xx}$ and

$$\phi_4 = -\frac{\phi_{2,xx}}{4\cdot 3} = +\frac{\phi_{0,xxxx}}{4!}. \tag{20}$$

Let us use $\varphi \equiv \phi_0$ as a shorthand, and so

$$\phi = \varphi - \frac{1}{2!}\,z^2\,\varphi'' + \frac{1}{4!}\,z^4\,\varphi^{(\mathrm{iv})} - ... \tag{21}$$

With this, we find

$$u_x = \phi_{,x} = \varphi' - \frac{1}{2!}\,z^2\,\varphi''' + \frac{1}{4!}\,z^4\,\varphi^{(\mathrm{v})} - ... \tag{22}$$

and

$$u_z = \phi_{,z} = -z\,\varphi'' + \frac{1}{3!}\,z^3\,\varphi^{(\mathrm{iv})} - ... \tag{23}$$

This has to be inserted back into the full time-dependent equation (2). Furthermore, to obey the ordering (1), we define $\epsilon = A/h$ for the amplitude A and $\delta = (h/\ell)^2$ for the height. Solving order by order, one arrives eventually at the equation

$$\zeta_{,t} + c\zeta_{,x} + \tfrac{3}{2}\frac{c}{h}\zeta\zeta_{,x} + \tfrac{1}{6}\,ch^2\,\zeta_{,xxx} = 0, \tag{24}$$

to lowest order. Here, $c = \sqrt{gh}$ is the wave speed, and $c\zeta_{,x}$ is just an advection term that can be removed by going into a comoving frame to obtain the KdV equation in its usual form. Higher derivatives would occur at higher order expansions.

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