Handout 15b: Intermittency in turbulence & shell models

Turbulence is intermittent, which means it is bursty. Mathematically, this means different moments scale *differently*. In turbulence, this applies to velocity *differences*, $\delta u_r = u(x) - u(x+r)$, not the actual velocities $u(x)$. We skip for now the fact that u is really a vector and demonstrate what it means in practice by studying a simple model of turbulence—a shell model!

1 Shell models

Shell models assume nearest (or next-nearest) neighbor coupling. Let us start with nearest neighbor coupling. So, a *toy model* of turbulence is the following:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -k_n \sum_{i=-1}^{1} \sum_{j=-1}^{1} C_{ij} u_{n-i} u_{n+1} - \nu k_n^2 u_n, \tag{1}$$

where, for further simplicity, we have assumed that u_n is real. The last term on the right corresponds to the dissipative term, and the the first one on the right is a quadratic nonlinearity, representing $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$.

The *most* important aspect in constraining the coefficients C_{ij} is the assumption that some particular conservation law should be obeyed. Energy conservation is obvious, and this means that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{1}{2} u_n^2\right) = -u_n k_n \sum_{i=-1}^{1} \sum_{j=-1}^{1} C_{ij} u_{n-i} u_{n+1} - \nu k_n^2 u_n^2. \tag{2}$$

By writing out all 9 terms, we see that there are only 2 pairs that can balance, so

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = A\left(-u_{n-1}^2 + 2u_n u_{n+1}\right) + B\left(-u_{n-1} u_n + 2u_{n+1}^2\right) - \nu k_n^2 u_n. \tag{3}$$

Remarkably, such a model reproduces the expected Kolmogorv scaling. For logarithmically spaced shells, the energy spectrum is $E(k_n) = u_n^2/k_n$. Furthermore, the energy in each shell fluctuates and exhibits intermittency!

Intermittency is linked to the property that the ζ_p deviate from a linear dependence on p. A completely non-intermittent behavior would mean $\zeta_p = p/3$. A phenomenological relation that describes the behavior observed in experiments and simulations is given by the She–Leveque relation She & Leveque (1994)

$$\zeta_p = \frac{p}{9} + C\left[1 - \left(1 - \frac{2/3}{C}\right)^{p/3}\right], \tag{4}$$

where C is interpreted as the co-dimension of the dissipative structures. For weakly compressible or incompressible turbulence the dissipative structures are one-dimensional tube-like structures, so the co-dimension is $C = 2$. Under compressible conditions the dissipative structures tend to become two-dimensional sheet-like structures, so $C = 1$, which is also borne out by simulations of highly supersonic turbulence. Sheet-like dissipative structures are also expected in hydromagnetic turbulence, where these structures correspond to current sheets. In that case one expects the same scaling as for supersonic turbulence.

References

She, Z.-S., & Leveque, E., "Universal scaling laws in fully developed turbulence," *Phys. Rev. Lett.* **72**, 336-339 (94).

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